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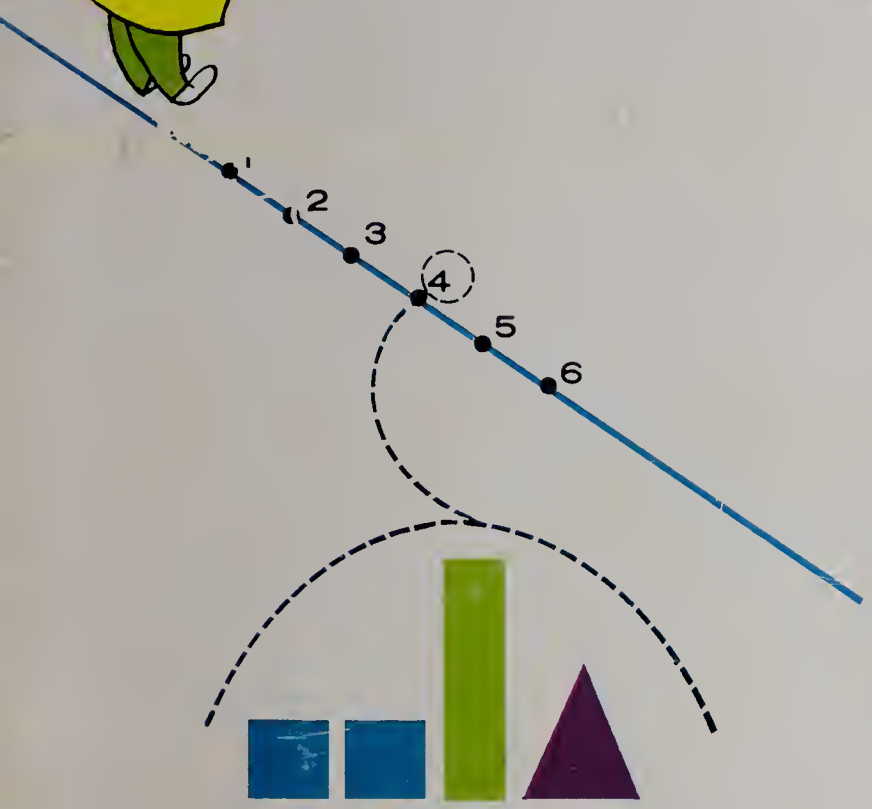

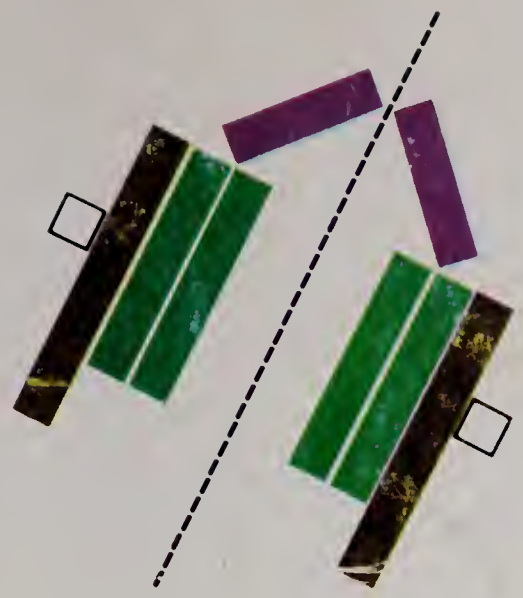

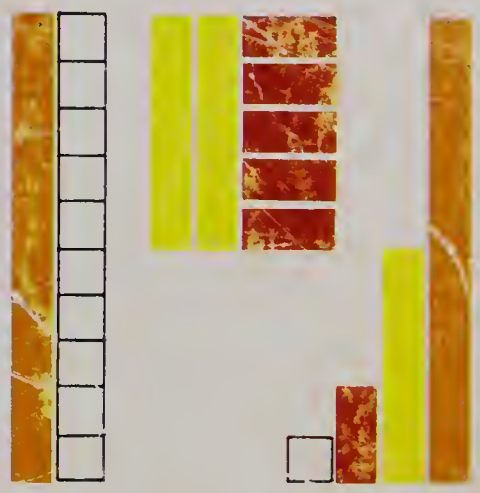

Games and Enrichment Activities

ATH WORKSHOP

■ Robert W. Wirtz ■ Morton Botel

■ Max Beberman ■ W. W. Sawyer

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Games and Enrichment Activities

MATH WORKSHOP

Robert W. Wirtz

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Project Editor

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Preface

This manual makes available to teachers a rich resource of practical classroom activities. It will be a valuable adjunct to whatever program teachers use.

The activities described in the manual involve the use of a wide variety of representational materials, devices, and models (homemade or commercially prepared).

Note that we have devoted a sizable block of pages to the use of the Cuisenaire® rods. The rods are one of the more versatile of the commercially prepared models. However, there are other materials which are readily available or are easily constructed and which can be used instead of the rods.

As teachers we recognize the fact that our pupils have a dual task in their arithmetic classes. They must *construct* their number concepts. They must also perform operations on numbers. Both the words "construct" and "operations" connote *actions*. For primary school children the actions should be performed on real things and preferably in game-like activities. Game-like activities are not necessarily competitive in nature. They are, rather, social situations wherein the teacher or pupils perform moves, countermoves, and other maneuvers which are governed by certain prescribed or agreed upon rules or procedures.

We are convinced that if in your daily lesson planning, you are guided by the contents of this manual, your pupils will build their own number concepts, become efficient in operating on numbers, and enjoy the quest for new knowledge of numbers.

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GAMES AND ACTIVITIES

NONVERBAL INSTRUCTION ACTIVITIES

The technique of nonverbal instruction was developed in response to a number of interesting questions, including: (1) How can we help teachers refrain from overexplaining things and permit pupils to make discoveries for themselves? (2) How can we increase the interest and attention span of pupils? (3) How can we teach universal mathematical concepts to children from another culture with whom we could not communicate verbally because of our different natural languages? (4) Would it be possible to teach mathematics if we had laryngitis? (5) How would we teach deaf children and nonreaders so that our activities would be geared toward their capacities rather than toward their limitations?

"Nonverbal instruction" is a term to be taken literally. It refers to group activities involving the entire class, in which practically no words are spoken or written by teachers or children. (Although we use the number line in the following development, many of the activities suggested in this manual can be taught through a nonverbal approach.) For example, suppose that, without introduction of any kind, you go to the chalkboard and start drawing a sketch or diagram. For reasons we don't fully understand, your pupils will be utterly silent. Their attention will be fixed steadily on the developing sketch.

At the appropriate time, you look for someone who can extend something he sees going on in the sketch—and the nonverbal activity is under way.

With no special training in this technique or in mathematics, any teacher can secure very wide and eager participation from most pupils. Children will not utter a sound—other than gasps of excitement, which often slip out when someone thinks he has discovered a pattern he wants to extend.

Activity 1

Draw a line on the board, as in the diagram:

Begin labeling the points this way, making no comment:

Now point to the fourth dot and get ready to label it. Stop. Look at the class. Offer the chalk to the class, thus suggesting that someone might want to continue the pattern. If there are no takers, go back and tap each of the tallies. First dot—one tap; second dot—two taps, etc. Then offer the chalk again. Repeat if necessary. Go on to label the fourth dot yourself if you need to. Soon you will have all the help you need.

Activity 2

Draw a line, labeling some of the points with numbers. (Use tags until pupils can write the numbers.)

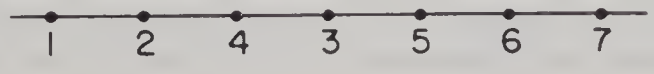
Now let pupils choose the tag that labels the next point, or have them write the appropriate number if they are able to. (Clotheslines with knots for points, wire with paper clips, or masking tape on the floor make good number lines. Large tags with paper clip hooks should be available. On one set, or on one side of a set of cards, draw tallies representing the numbers 1 to 10. On another set or side, write the numbers 1 to 10.)

Activity 3

Psychologists have long recognized the urge to complete the uncompleted. To look at an empty place where something ought to be is unsettling to the human mind. Children are particularly eager to make it right by filling in the missing parts.

Draw a number line on the board and label it. In labeling, there are at least two things you can do that will cry out for help, and there will be lots of helpers eager to get things back into comfortable equilibrium. One thing to do is leave a blank, as in the number line below:

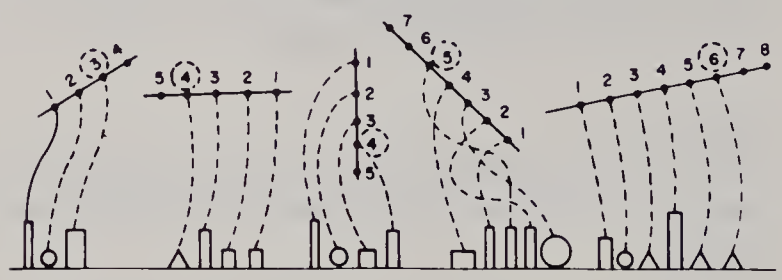
In the next number line, the naming of the points is wrong, and again there will be many helpers to make things right.



Activity 4

Draw several short number lines across the board at varying angles and with some numbered from right to left. Under each, on the chalkrail, arrange groups of objects such as rulers, circles, erasers, blocks, etc.

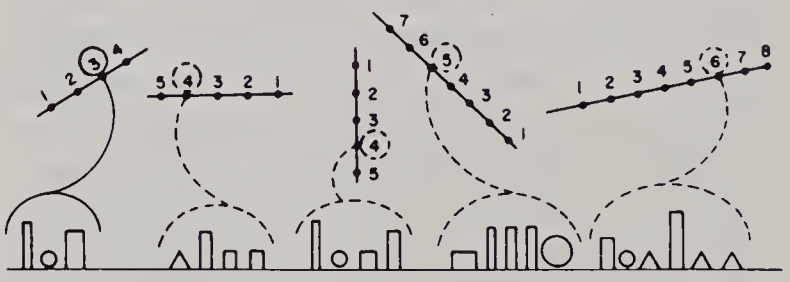
Draw a line connecting an object with the 1 on the number line above it—and that is all you will need to do. Everyone will want to get into the act.



When this becomes a familiar activity, it will proceed at a fast pace. Perhaps you can let the children line up and each draw a line. (It might be best to have them line up before you begin—since once you have lost your voice, you really ought to let it stay lost.)

Activity 5

When Activity 4 becomes familiar, erase all the connecting lines that were drawn, and start with the following kind of connection, letting pupils draw grouping and connecting lines for the other sets:



Erase the lines and move the objects into different groupings and look for volunteers.

Activities 4 and 5 are actually very different. In Activity 4 there is a one-to-one matching between objects and points on the number line, beginning with 1 and matching to each successive point. This reminds us of the blind giant Cyclops who felt the sheep as they left for pasture and took one stone out of a bag for each sheep. At night, when the sheep returned, he would feel them again and put one stone in the bag for each sheep. In this way he could tell whether or not they had all returned.

Activity 5 is much more sophisticated. Children must count a group of objects and select the number that is the answer to "How many?" Next, they must look for a symbol of that number and connect a line to the point on the number line that is labeled with that symbol.

It would be cruel to teachers and children alike to insist that these two ideas be explained in words. Communication would soon break down, and misunderstanding would become widespread.

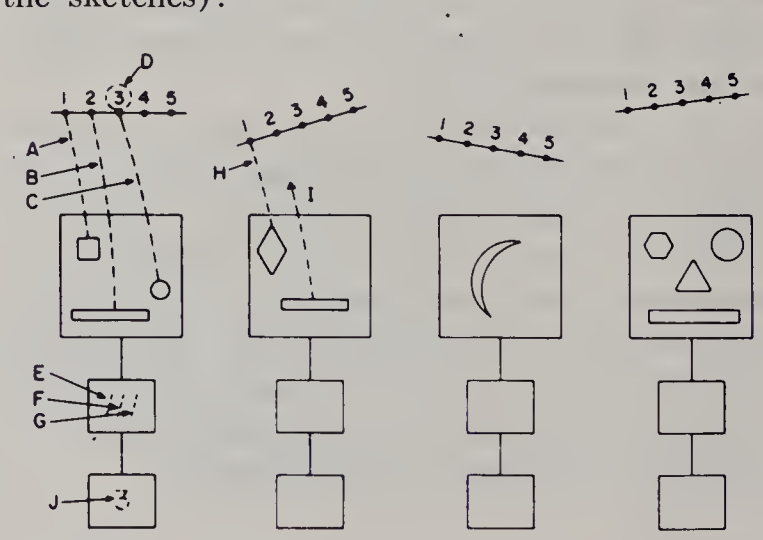
As a variation of Activity 5, leave the grouping lines and the connecting lines, but remove all the objects and place them in a pile on a table nearby. (This is a *reconstruction* activity.)

You look for help. What next? If no one volunteers, select a number of objects and put them back on the chalkrail under the grouping line that is connected to the point labeled with that number on the number line.

There will be many volunteers to reconstruct groupings, and there will be no concern about putting back the same objects in the same order and in the same sets. This is counting. Any collection of 5 objects is as good as any other collection of 5 objects to suggest the idea of the number 5.

Activity 6

Draw the following sketch on the blackboard (without the broken lines or letters which we use here so that we can refer to appropriate parts of the sketches):



While you draw number lines, boxes, and shapes, not a word is spoken. Then turn as if you are going to break the silence. Instead, turn back to the diagrams and slowly draw lines A, B, C, . . . but stop in the midst of drawing line I. (The letters in these diagrams are included so that we can refer

to them in this discussion. They should *not* be written on the board.) Turn to the class with a puzzled look on your face. Hands usually go up—but no one talks. Offer the chalk to an eager volunteer, who will usually continue line I to connect with the dot labeled 2 on the number line. Your smile of pride satisfies the volunteer and the class.

Then point to line D around the label 3 on the first line and then to 2 in which line I terminates. You want a pupil to circle that label. If someone guesses in a different direction and draws some other line, look at the new line, and erase it. If there are no volunteers, circle 2 yourself.

With your chalk, point from the square in the first diagram to line E—from the bar to line F—and from the circle to line G. Next, point to the diamond in the second diagram and to the box below it, then to the bar and to the same box. Now the puzzled look to the class—any volunteers?

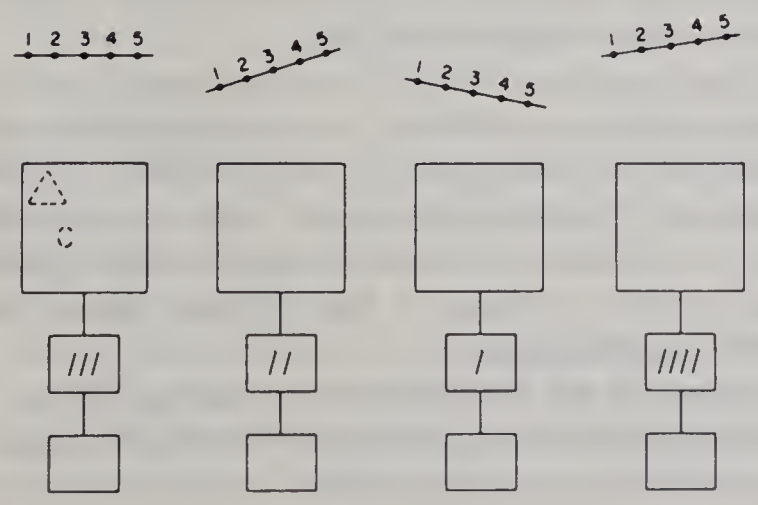
Usually a child will respond with some kind of tallies. If they are too dissimilar from lines E, F, and G, you may "fix up" the work. As the diagram grows, each success brings out a big smile from you.

When this is completed, except for the bottom row of boxes, point to the circled 3 on the first number line and to the box below the box of tally marks E, F, and G. If there are no volunteers, write a large 3 in that box—line J.

Then point to the 2 circled on the second number line and to the box at the bottom of the second sketch. There will be plenty of volunteers. If the numerals are not well formed, smile approval, and make a good model. Have the children complete the other two diagrams.

Activity 7

At this point, after admiring the work, erase much of it, leaving only the following skeleton:



Approach the first diagram. Touch one of the tallies and draw a triangle. Touch another tally and draw a letter C. Touch the last tally and, as you are about to draw something, turn to the class for volunteers, pointing to the last tally again and then to the section that already contains a triangle and a C. Of course, any drawing of a single shape or object is as good as another. If there is uncertainty, draw a tree or a house or a boy or whatever comes to mind.

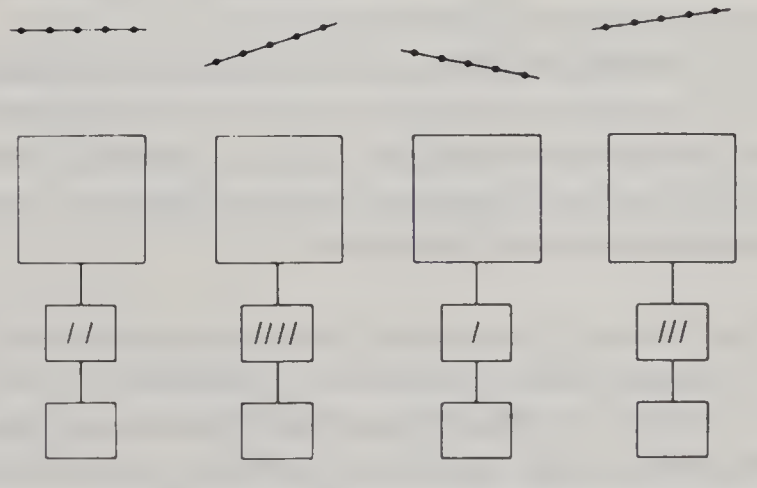
Admire the diagram and start a line from one of the objects toward the number line above. Stop and look toward the class. Now there are volunteers aplenty—and the set of sketches is reconstructed, except that the objects in each main box are different from those in Activity 6. The key to reconstruction is, of course, the number of tallies.

Activity 8

Erase again, removing the shapes in the squares, the connecting lines, and the tallies. Only the circles around the labels on the number lines are left as a key for reconstruction.

Another version can start with only the numbers in the bottom row of boxes as guides. Some children may not yet be familiar with those symbols, but their attention can be directed to one of the labels on the number line above, or they can get into the act as soon as a pupil has made the appropriate number of tally marks.

A switch may be made at this point to conclude the activity. Put in the tally marks—but in a different order.



Then a child adds a sketch to one of the big boxes. A smile of approval, and you reach out to take the chalk from the pupil—a signal that it is time for another volunteer. This is an essential procedure

by this time; otherwise some eager beaver would reconstruct the whole thing.

Several practically blank sketches similar to the one above can be prepared on a duplicator for those who would like to extend this activity by themselves.

In scores of experiments like this, the self-imposed silence has been maintained to the end. Accidents, of course, can happen, but they seldom do.

What Has Been Going On?

The primary idea we set out to develop is that the *count* or the *number* remains unchanged as the nature of the objects changes. Each pupil can draw a picture of anything he likes, because we are concerned only with the answer to the question, "How many?"

Another important idea is the one-to-one correspondence between a set of objects and the numbers of counting. Also, we have seen that the number assigned to a group is the last number matched as we move sequentially from 1 to 2 to 3, etc., in establishing this correspondence.

We have provided much opportunity for practice in recognizing the first four numbers of counting. Tallying may be a new idea—a most useful one for slower children, who otherwise might get lost as we move along.

The activity has individual parts in which every child can have an opportunity to succeed. At the same time, each episode is extensive enough to appeal to those brighter children who want to see ahead—who want to see the whole pattern. It certainly provides a rich mathematical experience, even to the newly initiated.

The model used here can, of course, be expanded to include larger numbers. Also, as we shall see, this model can be used to help the learner search for relationships between numbers. This is a very important part of arithmetic.

What Has the Non-verbal Technique Contributed?

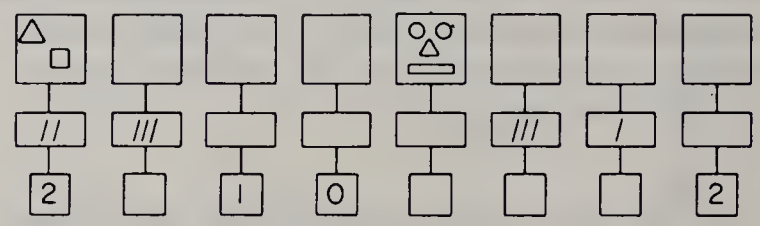
Anything that encourages self-discipline might be justified on that score alone. We do not urge this point. The pupils are certainly motivated; their interest does not wander—and this is true of almost every child. Most important, perhaps, is that we have been communicating with children without words. We are essentially "asking" children to do a series of things, sometimes in a certain order, always in a certain way and depending on given clues. If you were to put the instructions in words alone and limit them to the vocabulary of the class, you would be sorely taxed—and so would the pupils. You would feel that you needed to point out certain parts of the diagram to make your directions understandable. At this stage, ask yourself whether you might not give the clearest directions without any words at all.

A certain pattern is established in the first of the diagrams in Activity 6. This basic pattern varies only slightly as the entire activity proceeds. No explanation, however well remembered, will be nearly as useful to the child as a glimpse of the pattern. If no word is spoken, he must find part of the pattern if he is to participate. Some can grasp the whole pattern at once; others see it piece by piece and eventually put the whole together; some may never put the pieces together; and perhaps a few will see only a piece or two. Since mathematics is essentially a study of *patterns*, the technique that suggests this kind of explanation—the search for patterns—deserves much use by teachers.

Activity 9

Here is another activity that builds on the earlier ones. It has been unusually successful in first-grade classrooms.

Put the following sketches on the board:



Look quizzically at the board for a moment. Move toward the board as if to continue, but change your mind and look at the class.

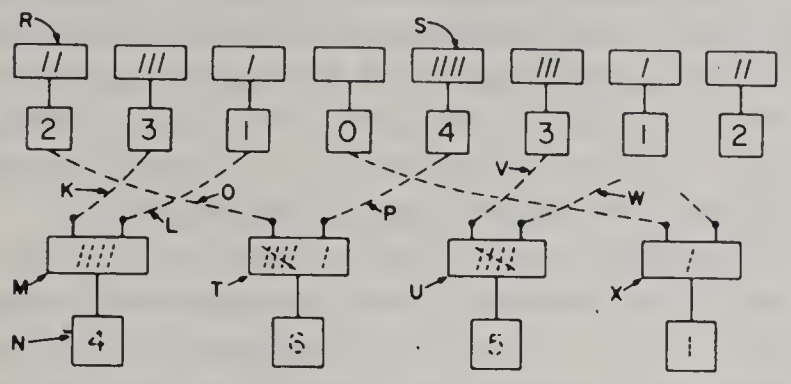
Hands go up. Offer the chalk to a pupil in the back of the room. Without a sound from the group, he comes up and goes to work on any of the sketches. His success wins your smile of approval, and now there are other volunteers. Often chalk can be given to three or four children who go to work simultaneously.

When all the diagrams are completed, the pictures in the top row and the numbers in the boxes in the bottom row are all erased.

With the tallies as clues, the diagrams are ready to be completed again.

Activity 10

Now erase all the pictures and boxes in the top row, leaving two rows—one of tallies and one of numbers. Add several pairs of boxes with pairs of antennae on top:



Draw connecting lines K and L. Point to the box to which these lines are connected. Look as if for volunteers, but don't recognize any waving hands. Point, one after the other, to the three tallies in the second diagram from the left, and then to the single tally in the box on its right. Point to them again—and then put four tallies in the box we labeled M.

Start to make a mark in the box we have labeled N, but stop and look around. There will be no lack of volunteers this time—to write 4 in that box.

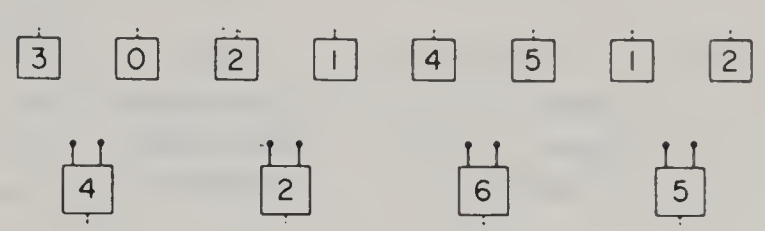
Next draw lines labeled O and P, and point to each of the two tally marks in the box marked R and then to each of the four tally marks in the box marked S. Several children are ready to take over, putting six tallies in box T and the number 6 in the box below.

If this is the first time you are using this pattern, complete the episode without introducing any new ideas. If, however, the time has come for a step forward, you may put five tallies in box U and draw connecting line V. Point to each tally in box U, hesitate, and point to the three tallies in the box above line V. Show disappointment that there aren't any more. Then begin to draw connecting line W—and stop. Any volunteers? You can be quite sure there will be.

Next, the box with a zero is connected to box X and one tally is drawn in X. The companion connecting line is started—to be completed by another volunteer.

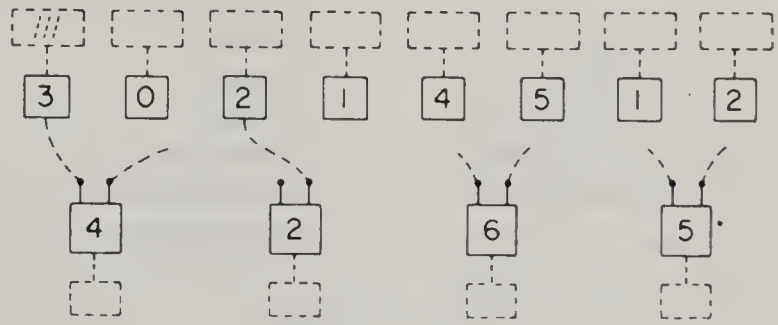
Activity 11

As soon as the children can write numbers, the tally marks can be omitted occasionally—but not often. Sketch the following diagram on the board and look for help:



There will be plenty of volunteers.

Thus, the ideas of addition and subtraction are planted. The idea of 0 has been suggested. In this example, a brief excursion has been taken into the world of abstraction in which we work with numbers alone. To tie this abstraction down quickly, relating it to counting as soon as it is completed, add the boxes in broken lines:



Put three tallies in the box above 3 (or sketches of objects). Children will finish the diagram, putting in all the tallies themselves.

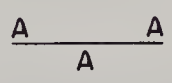
As if to check, tap each tally in the box below the 4 in the bottom row, and then the tallies in the boxes above the connecting lines. These taps will establish again, by sight and by sound, that there are the same number of tallies above and below the number. Do this with each part of the picture.

SPATIAL RELATIONS ACTIVITIES

A delightful group of games develops when we ask pupils questions that focus, at the same time, on how many *and* on where. In addition to the obvious emphasis on counting and the anticipation of addition, these games develop powers of at-

tention and increase the pupils' awareness of the relationship between locations of objects in space.

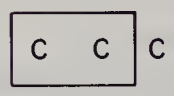
(1)

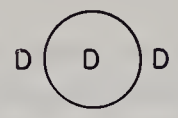


above
below
altogether

to the left
to the right
altogether

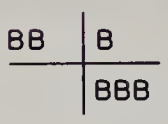
(2)



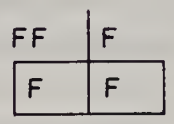


inside
outside
altogether

(3)

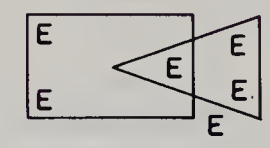


above
below
to the right
altogether



inside
outside
above
to the left
altogether

(4)



inside box
only inside box
inside triongle
only inside triongle

inside box and triongle
outside box
outside box and triongle
altogether

In talking about the sketches for these activities, you might ask the following kinds of questions:

1. How many letters (or A's or small a's, etc.) are above the line?
2. How many . . . are below the line?
3. How many . . . are there altogether?
4. How many . . . are to the right of the line?
5. How many . . . are to the left of the line?
6. How many . . . are in the box?
7. How many . . . are outside of the box?
8. How many . . . are . . .

(See the section on the And—Or Games on pages 28-29 *et seq.* and The Game of Fewest Questions on pages 30-31 *et seq.*)

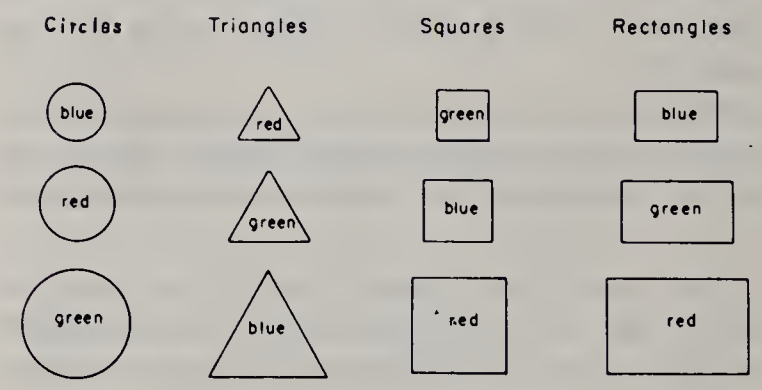
ACTIVITIES WITH GEOMETRIC SHAPES

Introduction

Approaching mathematics through the handling of objects permits each pupil to learn at his own best rate, with success for every child regardless of age or ability. At the same time, these activities offer unlimited opportunities to challenge those who can move swiftly to more advanced levels.

Manipulative materials frequently suggest important mathematical ideas more accurately than words. While searching for patterns, discussing them, and exploring their inner workings, pupils are learning how to think in mathematical terms.

Prepare, from construction paper, a group of three circles, three squares, three triangles, and three rectangles. Make one group for yourself and one for each pupil. Use the following pattern:



Such a collection makes possible many interesting activities.

Activity 1

This involves discussions of shape, color, and relative size. "Which are circles?" "Which circle is the smallest (the middle size, the largest)?" "What is its color?" "Give me the smallest square and the largest circle."

Activity 2

Set out an arrangement (on the chalkrail or felt board) similar to those that follow, asking pupils to arrange their own shapes in the same way on their desks:

a. ○ ◯ △ △ □ □

b. ○ △ □ ▭

c. ▭ □ △ ○

d. ○ △ □ ▭

By asking pupils to tell what the shapes are, beginning at the left, you are giving them practice in the prereading skill of left-to-right scanning.

Activity 3

Give pupils several copies of some of the shapes. Then place on the felt board various arrangements of one, two, three, four, etc., objects for pupils to reproduce at their seats:

for 1

△ or ○ or ◯ or □ etc.

for 2

△○ or △△ or □▭ etc.

for 3

○○ or △△□ or □△○ etc.

for 4

△○ □ or □□□□ or △ ○ ○ etc.

for 5

□□ □ □□ or ○○△□□ or ○○○ ○○ etc.

for 6

□□ □□ □□ or □□□ □□□ or △△ □□□ etc.

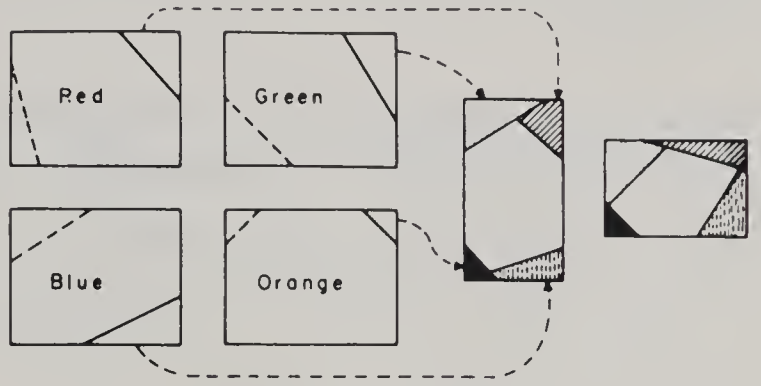
After pupils have learned to copy arrangements, you can make the game more interesting by increasing the challenge of the activity in three stages:

1. You cover the model after pupils have viewed it for a few seconds, and pupils reconstruct the model from memory.
2. Pupils copy the model with crayons.
3. Pupils draw the model from memory with crayons.

In a variation of Activity 3, ask pupils to show their own arrangements of two objects, three objects, four objects, etc.

Activity 4

Suppose that we had four pieces of colored paper —red, blue, green, and orange. We could cut one corner off each sheet, some big pieces and some smaller pieces, and paste these four corner pieces on a sheet of white paper, fitting the corners we cut off into the corners of the new sheet.



"Here is a picture of four sheets of colored paper. Who can draw a line on the sheet marked red to show a corner we can cut off?" Ask the same question for each sheet. Urge some children to make big pieces, small pieces, long, thin pieces, etc.

"Now, let's look at the first picture frame. It's empty. What would it look like if we pasted that red corner on the picture, fitting it neatly in the corner?" Do the same with the other colors.

Repeat the activity, starting with a line that indicates another corner cut off the red piece. Construct a second design.

Let the children repeat the activity, working either individually or in small groups, preferably no more than four to a group. Each group needs several pieces of colored paper. Each child is to make two designs, cutting off corners of the colored paper and fitting them in the corners of the white paper.

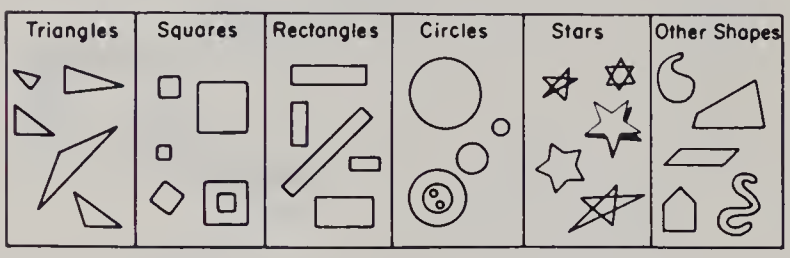
(The corners with which the children are working will later assume the dignified name of *right triangles*. The idea of *fitting*, as it is used here, becomes very important when we want to talk about *angles of the same size*, a condition which exists when one angle can be fitted into the other.

When the three angles of a triangle can be matched with the three angles of a second triangle so that each of the angles of the first triangle can be *fitted* into the corresponding angle of the second triangle, we call the triangles *similar*. The idea of angles fitting or not fitting is a necessary concept when comparing shapes.)

Paste-up design work similar to that in the preceding notes should be developed around other shapes.

Activity 5

Section the board and write a name in each—Triangles, Circles, Squares, Rectangles, Stars, Other Shapes.



"This section is for triangles. Who wants to draw a triangle in this section? Another triangle? A long thin triangle? A small triangle?"

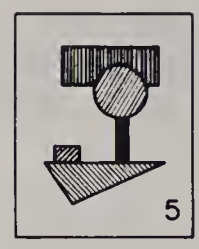
Then move on to squares. Bigger squares, smaller squares, a square inside of a square, etc.

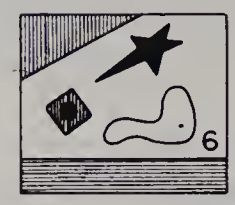
Whenever a volunteer draws a shape that doesn't belong, talk about it. "Does everyone think this shape is in the right place? Where should it go?" If it can't go in any other specific section, it can always go in the section titled "Other Shapes." No contribution need be excluded; some merely have to be redrawn in another section.

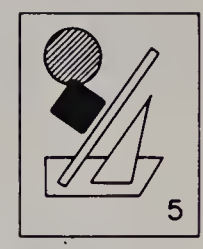
Now ask each child to draw on paper at least one shape for every section—a triangle, square, etc. Then ask the pupils to cut them out. Encourage the children to use paper of various colors and to see if they can use up all of their scrap pieces.

"Pick five of the shapes you have cut out. Paste them on a sheet to make a design you like. Then take six more shapes and make another design. In the right-hand corner at the bottom, write the number that tells how many shapes there are in the picture."

On the board, sketch examples; or have some of the children sketch them. (Use several different colors of chalk.)







Have each child select a design he has made, the one he likes best. Line them up on the chalkrail or on a bulletin board. Give each a letter or number designation.

Tell the pupils that they are a jury. The job of each child is to study all the designs and pick the three he likes best. He is asked to write the letters or numbers of these three on a piece of paper and put it in a ballot box.

Should you discuss the three or four or five selected designs? Very briefly and not in detail, if at all. Children are good art critics but they are not able to express their reasons for preferring one design to another. They respond to balance, symmetry, movement, color harmonies, etc., without having the background to verbalize such reactions.

It is enough for the children to have participated in selecting favorites from a whole collection of abstract designs. These are not pictures of people or trees or cars that look like people or trees or cars to everyone. The children have made a judgment. *Likeness* was not the basis of judgment. What was the basis? You will never know, but it will be a mixture of balance, symmetry, movement, color combinations, and all the qualities that an art critic can verbalize.

PATTERN ACTIVITIES

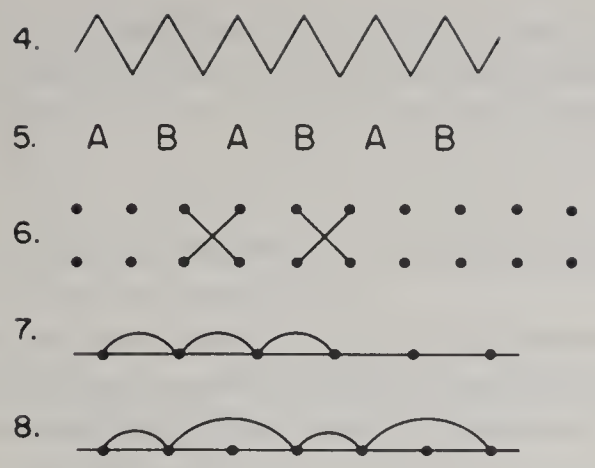
Recognition of a pattern, or of "What's going on?" is a most basic activity in mathematics. Patterns are of many kinds—color, order, shape, size, etc.

The following kinds of activities will help to develop sensitivity to this idea of pattern.

1. Have pupils string beads or draw pictures, beginning with such simple patterns as red, blue, red, blue (R-B-R-B).

2. ○ △ ○ △ ○ △

3. — + — + — +



2. Children can line up in front of the rest of the class and make up patterns: boy, girl, boy, girl, etc. Or, "Every other one raise your right hand. Others raise your left hand."

Each of the above patterns is open on two ends. Each can be extended to the right and to the left, and can be developed through an unlimited range of complexity.

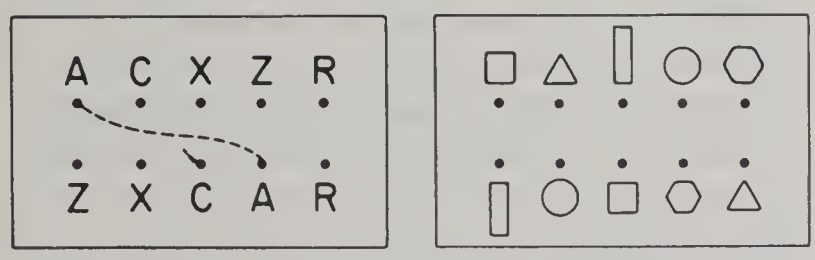
MATCHING ACTIVITIES

Introduction

In the following matching activities, examples are given of the kinds of work you may give your pupils. Many pupils will not get beyond exercises like those suggested in Activities 1, 2, and 3 for a long time.

Activity 1

You have laryngitis. Put examples such as the following on the board:

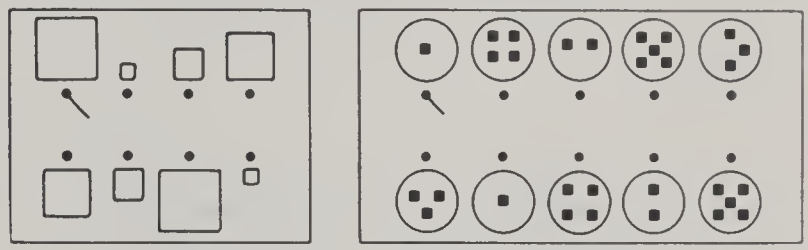


Start drawing a line from the dot under the top A and stop. You don't have to complete the line. Look around for volunteers. There are probably several ready to help. Hold out the chalk to one. If he continues the line you started until it connects to the dot over A in the bottom row, you beam from ear to ear because your mental telepathy was successful.

If the volunteer does anything else, you are upset because your telepathic effort wasn't well enough carried out. Remember, you have not said a word; no one knows the rules for sure; no one can make a mistake. You have to try again and again until you have *communicated.*

Start a line from the dot over the C in the bottom row. Again decide against completing this job and look for volunteers. There will probably be many by now, and you can move rapidly through both of the matching examples.

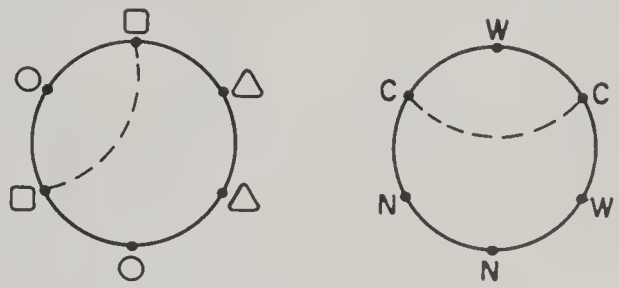
Activity 2



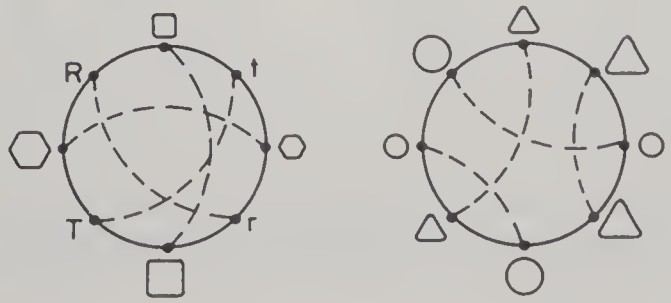
These examples are not difficult and go very fast.

Activity 3

A similar activity, a bit more sophisticated, develops when the objects or pictures to be matched are not displayed in two rows, but rather around a circle:

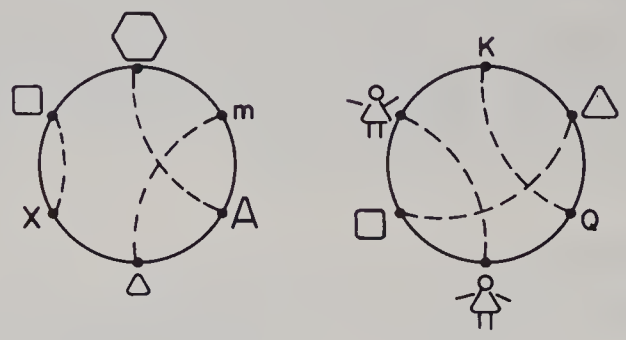


Then more matching with more sophistication. (In the second example below, there are four of each kind of symbol, so the connections can be made in several ways, any of which is as good as any other.)

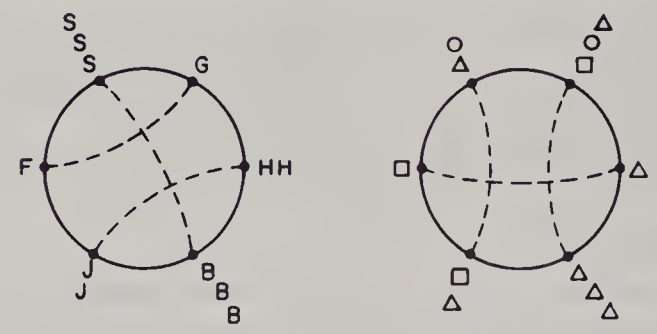


Activity 4

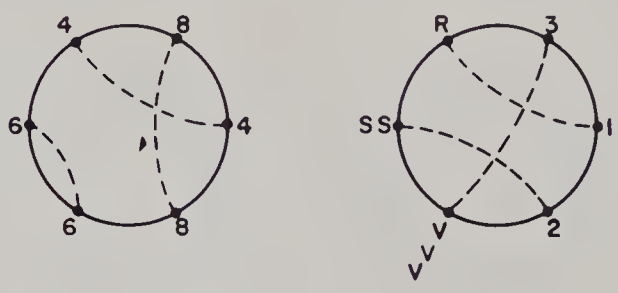
You still have laryngitis. In the first example below, largest, smallest, and in-between are the basis of matching. In the second, letters, shapes, and people.



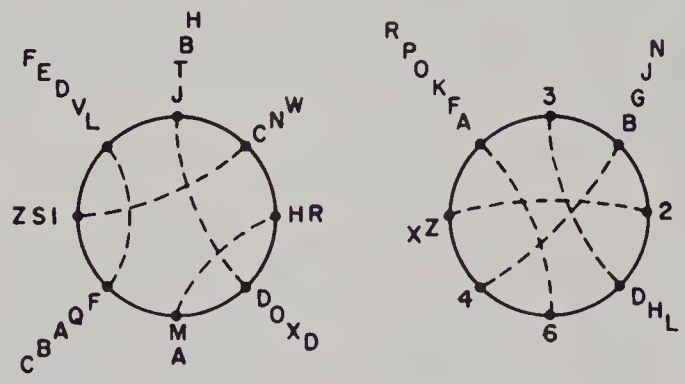
Activity 5



And on to the following:



You still haven't said a word, but activity is going ahead rapidly:



We have come a long way in a short time, moving through several very different kinds of matching, changing rules as we went—all without a word. There are no words that would convey each succeeding idea as well as letting children observe and conjecture.

THREE BOXES—A THINKING GAME

Three Boxes is a thinking game that can be played with any number of boxes, but playing with less than three is a bit limiting, while more than three can easily become too difficult to talk about.

Let's start with one box on a desk. It is empty. You have a piece of chalk in your hand. You stand in front of the box and put the chalk into the box. Then you stand aside and ask, "Can you tell me anything about the box and the piece of chalk?" Of course. There is a piece of chalk in the box.

Now, start again with two boxes and one piece of chalk. Stand so that your pupils can't see what you do with this piece of chalk. Put it in a box. Stand aside and ask, "Can you tell me anything about the boxes and the piece of chalk?" Certainly. There is a piece of chalk in one of the boxes, no chalk in the other. There isn't more than one piece of chalk in either box. If the chalk is in the box on the right (or the blue box, etc.) then there is none in the box on the left (or the red box).

Start again, this time with two pieces of chalk. Stand so pupils can't see what you do. Put a piece of chalk in each box. Stand aside. "Can you tell me anything about the two boxes and the two pieces of chalk?" Someone may say, "There is one piece of chalk in each box." You respond, "Are you sure?" Let discussion develop.

"Can you tell me anything about the two boxes and two pieces of chalk and be *sure* that it is true?" Certainly. If there is only one piece of chalk in the red box, then there is one piece in the blue box. If there are no pieces in the red box, then there are two pieces in the blue box. There are not more than two pieces in either box. There is at least one piece of chalk in one of the boxes.

Start over again. Put three pieces of chalk in the two boxes. "What can you say now and be sure that what you say is true?" There will be many answers, one of which should not be overlooked. If it is not given, you ask, "Can you be sure that there are at least two pieces in one of the boxes?" The answer is "Yes," but it takes some reasoning that isn't obvious to children (and some adults).

Now we are ready for the three-box version. Start with one piece of chalk. There is one box in which there is a piece of chalk, and the others are empty. There cannot be two pieces in any box, etc.

Next, use two pieces of chalk. There is at least one empty box. If there are two pieces in one box, then there are two empty boxes.

Then use three pieces of chalk. If there are not two pieces in at least one box, then there is one piece in each box. (Use boxes of different colors so that a child may answer, "If there are only two pieces in the blue box, then the green and red boxes cannot both be empty and they cannot both have pieces of chalk.")

Then use four pieces of chalk with the three boxes. "Can you be sure there is at least one piece in every box?" (No.) "Can you be sure there are at least two pieces of chalk in one box?" (Yes.) This again is a question that requires a bit of reasoning. There are many responses children can make; for example, "If there are two pieces in each of two boxes, then the other box is empty."

Finally, use five pieces of chalk with the three boxes. If one box is empty, then there are at least three pieces of chalk in one of the other boxes. "Can we be sure there are not at least two pieces in each box?" (Yes, we would need at least six for that to be true.) "If there are two pieces in one box, can we be sure there are at least two in one of the other boxes?" (Yes.)

You can play this game as many times as the children like. With it, they learn to reason mathematically. Each "If . . . , then . . ." statement is the end product of logical reasoning. The game is also an exercise in precise expression. The game is no fun unless we are sure that everyone says exactly what he means. This gives you, the teacher, an opportunity to emphasize the need, in certain circumstances, for precise expression.

FINGER PLAY—A THINKING GAME

Our decimal system is usually considered conclusive evidence that early mankind used fingers as an aid to counting. Most children are fascinated by their fingers. Many can count their fingers before they enter school. Some will employ this built-in computer aid until it becomes inadequate.

There is no point in avoiding a natural use of the fingers in counting and computing but, at the same time, experience has revealed that we do not always understand the way in which children use their fingers as aids. For example, if you hold up five fingers on one hand and one finger on the other and ask, "How many?" there is usually a ready response—"Six." But then hold up four fingers on one hand and two fingers on the other. Response is slow and uncertain. Children know they have five fingers on one hand, and that six comes after five. However, they are confused by this different arrangement of six fingers. This suggests that we can develop some finger-playing techniques that will broaden the way in which fingers can be used to suggest small numbers.

The Game

"The rule today is—everybody keep both thumbs down. Put your hands in your lap. Fix your fingers on your right hand so that three are up. Ready? Hold up your right hand. How many fingers are up? How many fingers on your right hand are down, counting your thumbs?

"Hands in lap again. Fix your fingers so that there are five up, but no thumbs, remember. Can you do it with one hand this time? Then use both hands. Fix five fingers up and get ready to show them. Hold up your hands. How many fingers are up? How many fingers are down, counting your thumbs?

"Hands in lap. Fix your fingers so that there are four fingers up . . . some on each hand . . . but no thumbs! Fix four fingers up . . . some on each hand. Ready! Hold up your hands. How many fingers up? How many fingers down?

"Hands in lap. Fix your fingers so that thumbs and little fingers are down and all the rest up. Ready? Hold up your hands. How many fingers are up on your right hand? How many up altogether? How many fingers are down on your right hand? How many are down altogether on both hands? How many little fingers are down? How many thumbs are up?"

(Whenever you can ask questions for which the answer is, "There are none," you have given further reinforcement to the idea of 0. Examples of such questions are: "How many pennies are in an empty box?" "If you have only two blocks on the table and somebody takes them off, how many blocks are left?")

"Hands in lap. Fix your fingers so that six are up, but no thumbs. Fix your fingers so that there are more up on your right hand than on your left hand. Remember—no thumbs. Six fingers up altogether —more fingers on your right hand. Ready? Hold your hands up. How many fingers are up altogether? Are there more up on your right hand than on your left hand? How many thumbs are up? How many fingers are down (including thumbs)?

"Hands in lap. No thumbs. Fix five fingers up with the same number on each hand. Ready?" (You hope not. This always leads to looks of puzzlement and often a healthy, "You can't do it!") "That's right, we can't do that.

"Let's try another. Fix five fingers up but no thumbs and all on your right hand. Ready?" ("You can't do it!" comes out more readily this time.) "We can't do that one, either, can we?

"Let's try fixing five fingers up this time using both hands—no thumbs—only one finger on your right hand. Ready? Hold your thumbs up. How many fingers are up? How many fingers are down?"

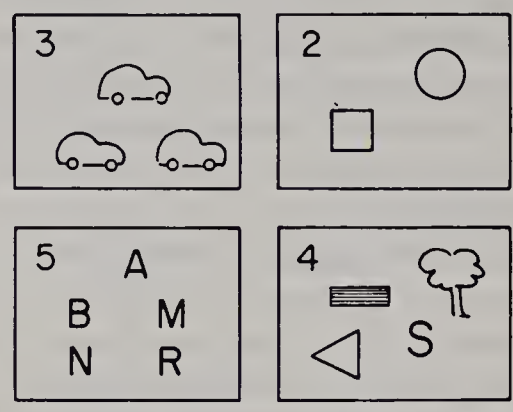
Occasional finger playing along these lines helps develop the idea of numbers, increases coordination, expands vocabulary, and is fun. You can match your instructions to the level of maturity of the children. After they have become familiar with the activity, let some of the children call out directions.

The thumbs need not always be excluded, but should be excluded often enough to let the children see eight as two groups of four as well as three more than five. Some day you might exclude the little finger on the right hand and the thumb on the left hand.

By having the children start with their hands in their laps and fixing their fingers before showing them, you can detect more easily those who are having trouble. Everyone ought to be able to respond if given enough time. Repeating the instructions several times when the activity is new helps give a feeling of confidence to those who are cautious or slow or timid.

PICTURES SUGGEST NUMBERS

For seat work, have children fold a piece of paper so that, when opened, it is divided into four, six, or eight rectangles. Have them draw a line over each of the creases. Then, on the board, write a number in the upper left-hand corner of a rectangle—3, for example. Ask a child to draw three objects or designs in the rectangle (they need not be alike). In other rectangles, begin with one of the numbers 2, 5, or 4 in the upper left-hand corner. Children then draw appropriate pictures. The board might look something like this:



"In the space where we drew a picture of cars, we also wrote 3. Is it in the center of the space? Where is it?" (It is in a corner.) "How many corners are there in that space or picture? How many corners at the top? How many at the bottom? How many on the right-hand side? How many on the left-hand side? Where is the bottom right-hand corner? Where is the top right-hand corner? Did we write 3 in the top right-hand corner? Where did we write 3? Where did we write 2 . . . 4 . . . 5?"

Ask the children to look at the paper they prepared. "How many spaces are there? Could you draw a picture in each space? How many corners does each space have? How many top corners in each space? How many bottom corners? How many corners on the right-hand side? How many on the left-hand side?

"Put your finger on the top left-hand corner of one of the spaces. Think of your favorite number." (You may need to say, "less than 10.") "Now, write that number where you had your finger. Write it in the upper or top left-hand corner of one of the spaces." Proceed until each space contains a number in the upper left-hand corner.

"Now draw a picture or design that suggests the number you have written. Do this in each space."

When some of the children finish, you can ask them to try larger numbers on the other side of the sheet. Or you may wish to proceed with the activity as follows:

Draw attention once again to the board. "Here

is a picture of three cars. The number 3 is written in the upper or top left-hand corner. Where is the lower or bottom right-hand corner? Show us the bottom right-hand corner of each space or picture.

"There are three cars. If we draw another car in the picture, how many would there be?" (Have a child write 4 in the bottom right-hand corner of the picture.) "If we had one more design in the next picture, how many would this be?" (Have a child write 3 in the bottom right-hand corner and 6 and 5 in the others.)

"Cross out the numbers in the upper left-hand corner.

"Do you want to change the pictures now?" They will want to draw another car in the first picture, and add an item in each picture.

Now back to the children's pictures:

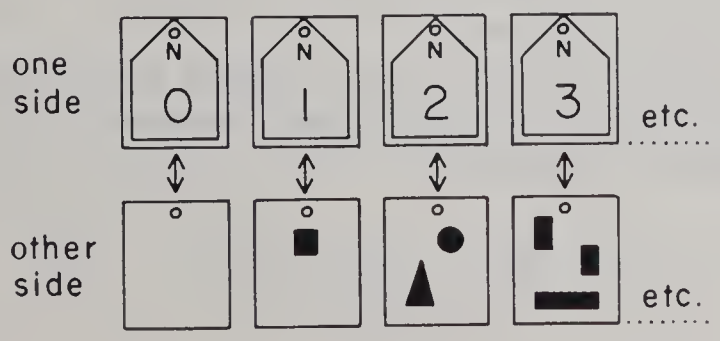
"In each space write a number in the lower right-hand corner that is one more than the first number you wrote in that space. After you have written the numbers in the lower right-hand corner of each picture or space, change your picture as the new number suggests."

This kind of activity can call for two more or three more, etc. If a child cannot write numbers, develop the activity, using a number in the upper left-hand corner and tally marks in the upper right-hand corner.

One aspect that children seem to enjoy is that they can select their favorite numbers and draw any objects or designs. The more enterprising will favor larger numbers; those who enjoy drawing may tend toward elaborate pictures; those who don't like to draw pictures can make dots or circles or simply lines.

COLLECTIONS AND LABELS

Make a set of cards about 6″ x 9″. Draw a tag on one side with one of the numbers from 0 through 10 (or more); on the other, draw the appropriate number of objects:

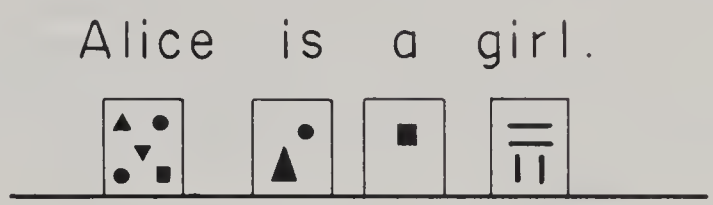


The N indicates that we are talking about a number—the answer to "How many?"

Arrange some cards at random on the chalkboard.

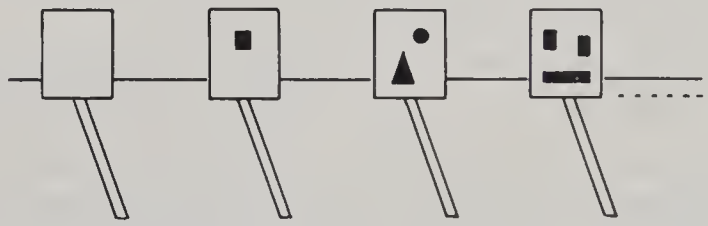
"Would someone like to change the arrangement? Have you a reason to like your new arangement? With the empty card on the left, can we arrange the cards so that every other card has 1 more picture on it than the one on its left?"

Write a sentence on the board. Under each word,



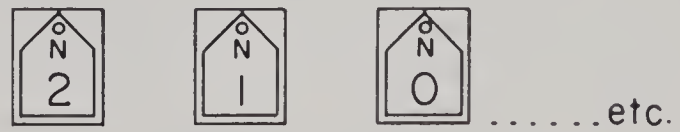
place a card with the same number of designs as there are letters in the word. (You may need several cards with the same number of objects.)

Arrange the cards in order on the floor and against the wall—about 15 inches apart. Running out from each, put a piece of masking tape on the floor. Result—a forerunner of the number line:



Point out that "a jump" will mean jumping from one piece of tape to the next. Start at the empty card and take three jumps. How many designs on the card by the final landing point?

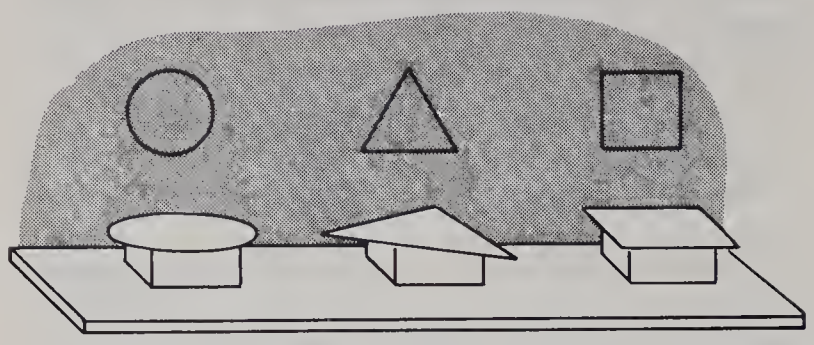
Displaying the other side of the card—with tags and numbers—an appropriate question is:



"How many designs will we find on the other side of this card?"

Now, each of the activities developed using the design sides can be repeated with the number sides.

A REGROUPING GAME



Assemble several boxes and pieces of cardboard cut in a variety of shapes—a circle, a triangle, a square, a hexagon, a rectangle, etc. Rearrange the boxes on a table pushed against a chalkboard as shown above. On the board draw a shape similar to each cover as indicated in the sketch. Count out some objects—such as 3 erasers—and put them in the box on the left (the one shown above with the round cover).

Ask a child to lift the cover and count the objects. When he announces that there are 3 erasers in the box, write 3 in the circle on the board:

The children are asked to close their eyes. They are told that all 3 erasers are going to be taken out of the box on the left, put in the other boxes, and that the covers will be put in place.

"Open your eyes." Ask someone to look in the box with the round cover. "There were 3 erasers in there before. How many now?" None, of course.

Someone can look in one of the other boxes. He might find 2 erasers in the box with the square cover. Note this information on the board in this way:

Question: "How many are there in the other box? Do you need to look? Can you be sure?" (One way to be sure is to lift the cover and count.) "Yes, only 1." Record the result:

Then write the symbols we use to express the idea suggested:

We say, "Three equals (or *is*) two plus one," and at the side, write:

$$3 = 1 + 2$$

Start this activity again with the same three objects or three other objects. One of the children may be asked to play the role of the teacher, moving the objects from the round-lidded box while the others' eyes are closed. When only one lid of the other boxes is lifted and the objects are counted, the question is, "How many are there in the other box?" The result is recorded in the list at the side. You might get the following record:

$3 = 1 + 2$
$3 = 0 + 3$

"Are there other ways three objects can be arranged in the two boxes on the right?" If there is a volunteer, he plays the role of teacher—moving the objects and recording:

$3 = 1 + 2$
$3 = 0 + 3$
$3 = 3 + 0$

"Any more ways that are different?"

$3 = 1 + 2$
$3 = 0 + 3$
$3 = 3 + 0$
$3 = 2 + 1$

If a child repeats one of the previous arrangements, the record of results is used to point that out.

A similar development can center around other numbers of objects.

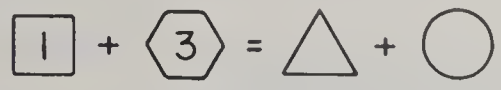
Vary the game. Put some objects in the boxes with the triangular and square covers. Count and record, for example:

"Close your eyes. Now I'm going to put all the objects from both boxes into the box on the left. How many are there in that box? Can you be sure?"

Repeat this activity with a shift in the order of the boxes—and a shift of the symbols in the results, for example:

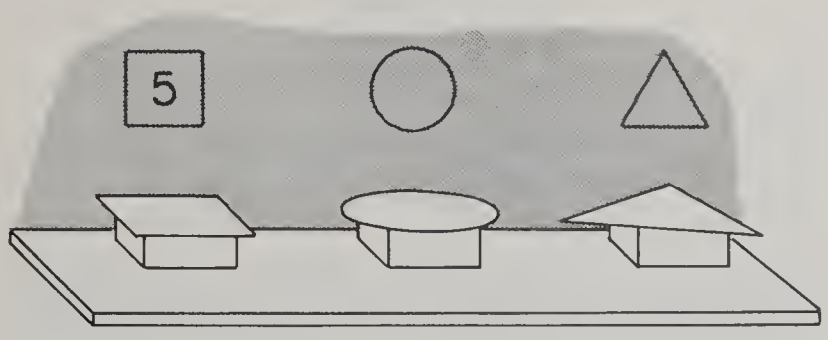
In each instance, when the children's eyes are closed, the objects are moved from the box or boxes on one side of the "=" to the box or boxes on the other side.

Try using four boxes:



After the objects are moved from both boxes on the left, lift a lid of a box on the right and count the number of objects in it. Try to anticipate the number of objects in the other box on the right.

Arrange three boxes as shown below. Count out some objects and place them in the box on the left. Note the number of objects in the appropriate place on the board:



While the children watch, take out some objects from the box on the left—2, for example—and put them in the middle box. Record this step:

Have the children close their eyes. "Now I'm going to move the objects left in the first box and place them in the third. Open your eyes. How many are in the box with the triangular cover?" Note the results. Then write:

5 - 2 = 3

and read, "Five minus two equals (or *is*) three."

Begin with a number of objects in the left-hand box unknown to the child. Move some—3, for example—into the second box. Record what the children know:

"Close your eyes. I'm going to move all that remain in the box on the left to the box on the right. Open your eyes and count the objects in the box on the right." Suppose that there is only 1. Record what the children know:

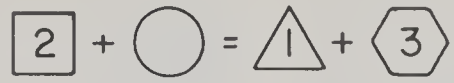
- 3 = 1

Question: "How many were there in the box on the left when this game started?" Complete the record.

"Here are parts of records of games. What did the whole record look like in each case?"

4 = + 1

5 + 2 =



2 + = 1 + 3

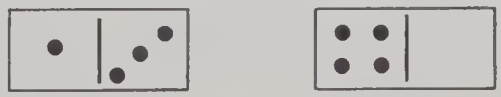
4 - 4 =

Act out any examples if there is any question at all. We can always find the results in arithmetic by counting objects in appropriate arrangements.

DOMINO CARDS

Consider making a set of double-sided domino cards—large enough to be seen from all parts of the room when the cards are set on the chalkrail. (About 2 feet by 1 foot with a heavy line marking each side into two 1 foot by 1 foot regions.)

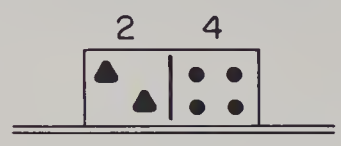
The designs may represent different numbers of the same object or of different objects. One card might look something like this:



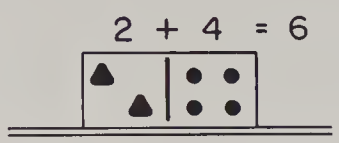
Another might look like this:

Activity 1

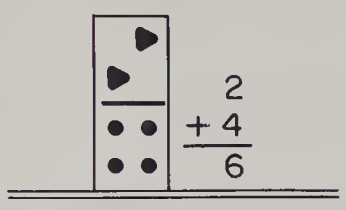
Place a card on the chalkrail, and ask about the number of objects suggested by each section of the card. On the chalkboard, close to each section counted, note the results:



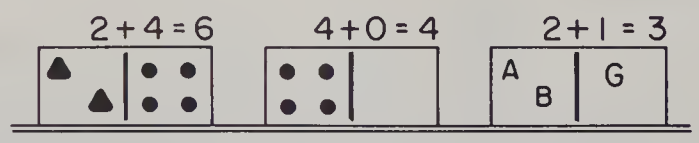
The next question is: "How many altogether?" Note the result and write the mathematical statement:



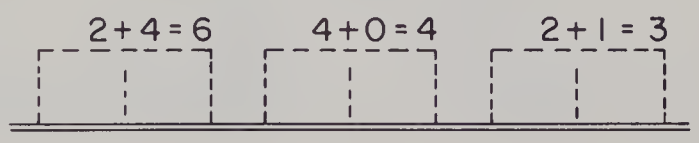
With the card in a vertical position, the complete record might look like this:



Leaving the card and record as they are, complete several more examples:

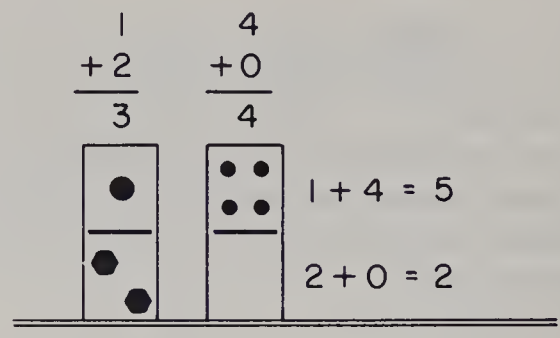


Now put the cards back in the general stack of cards. Ask pupils to pick out cards that fit each statement that is left on the board and put it in place on the chalkrail:



Activity 2

An extension of this activity involves the use of two cards at a time placed side by side on the chalkrail. Children are asked to consider each row and each column—noting the results in some such fashion as this:

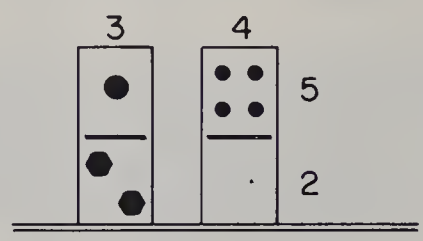


Remove the cards, and ask a volunteer to put them back in proper position.

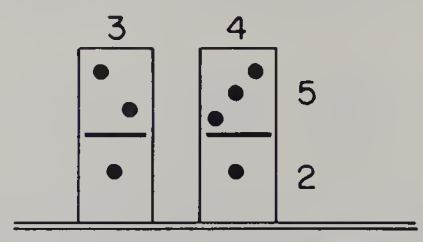
Leave the results on the board while developing other examples.

Activity 3

Another method of noting results in this last activity leads to another problem:

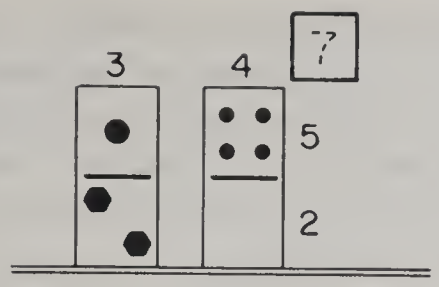


Now return the cards to the stack. Let the pupils discover that these same results may be suggested by different pairs of cards. For example:



Some pupils might be interested in finding all the pairs of cards in the stack that are appropriate.

Draw a box in the upper right-hand corner, and in it write the total number of objects shown on the cards:

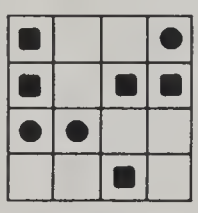


The fact that the sum of the top sums (3 and 4) is 7 and that 7 is also the sum of the side sums (5 and 2) is of fundamental importance. It illustrates the principle that the particular order and grouping used in adding numbers does not affect the total.

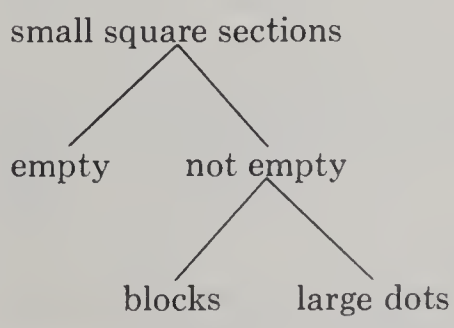
$$(1+2)+(4+0)=7$$
$$(1+4)+(2+0)=7$$

YOU SEE—I SEE

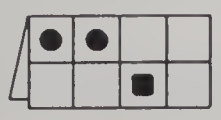
Below is a sketch of one of many similar pieces that can be used in a game that keeps children alert. It can be made of oilcloth or other flexible material.



Show the entire piece to the children. Notice that it is divided into a certain number of small square sections. Some of the square sections are empty. Some are not empty. Of those that are not empty, some contain blocks while others contain large dots. A diagram depicting the above information looks like this:



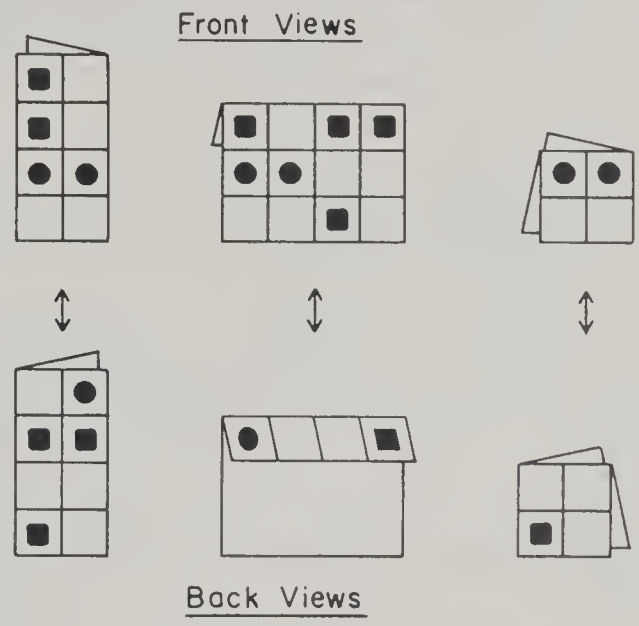
Fold as indicated below so children see only one side:



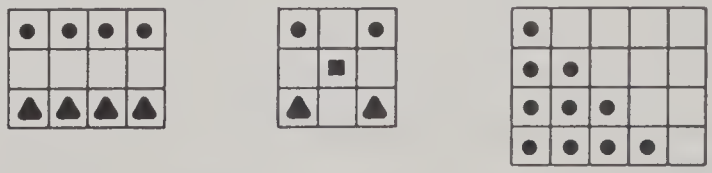
Now comes the switch. The teacher is holding the piece between himself and the class. The teacher looks at the side toward himself—the other side. Remind the children that they saw the whole piece in the beginning. "How many big dots do I see? . . . How many blocks? . . . How many empty squares?" Few, if any, will remember. Turn it around so the children see the other side. Repeat the questions, counting when necessary.

"Now, how many big dots do I see?" The teacher is now looking at the side away from the children—the side the class saw earlier.

While this activity starts out as a guessing game, it soon becomes a remembering game. As soon as most of the answers are unanimous, fold the piece differently—and repeat the entire activity. Here are three suggestions for folding:



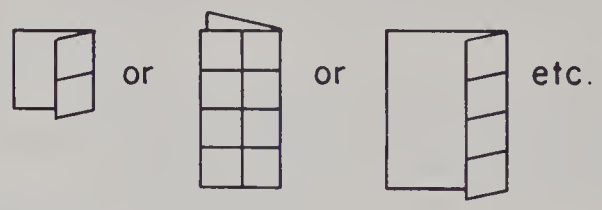
Make a set of similar pieces in different sizes with a variety of shapes or designs, including some examples that have some kind of symmetry or a reference point that children can use to help them remember or reconstruct. Here are some suggestions:



Note that each is a rectangular array. Children early begin to remember that the 4 by 4 examples contain 16 small squares; that when it's folded in half, there are 2 rows of 4 each, or a total of 8 small squares. They soon can tell at a glance that

there are 9 small squares on the 3 by 3 piece, etc.

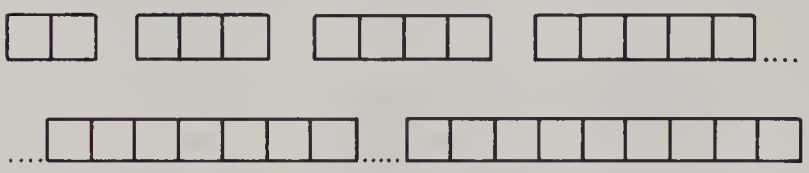
Another game to nurture the idea of multiplication develops from limiting the attention to "how many small squares," as in the following examples:



"How many small squares do you see? . . . How many do I see? . . . How many are there on the whole piece?"

Each of the activities may be organized as a game played by two groups—one sitting so they can see one side, and the other looking at the other side. A member of team A is asked to tell what team B sees; a member of team B is asked to tell what team A sees. Then the piece is folded differently and another round begins. Each team gets a point for each correct answer.

Another version of "You See—I See" can be based on a series of strips of oilcloth or other flexible material of different lengths:



Count the number of squares on a selected strip. Fold it:

"Now how many do you see? . . . How many do I see?" . . . (A subtractive situation.)

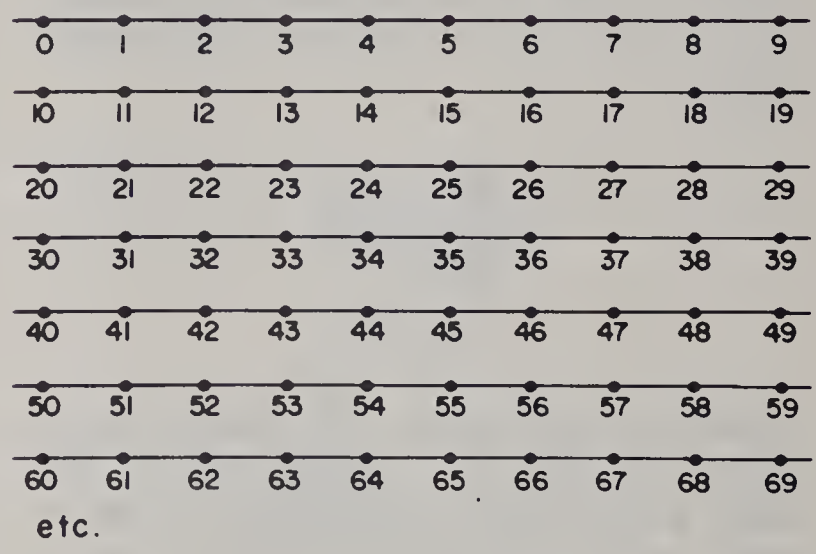
Pick a strip and fold it without counting the number. "I see 7 squares. How many are there on the strip altogether?" (Children need to add the number of squares they see to the number the teacher sees—an additive situation.)

Another version using this set is to fold a piece in half:

"How many are there altogether?" In this case there are 11. The pieces with an even number of squares develop the idea of twice as many; the pieces with an odd number of squares present a problem that involves the idea of fractions; you can only see half of one of the small squares, and this idea comes most naturally into the discussion. Few children, when they see:

will fail to know that this is the piece with 3 squares, though they will seldom answer that "one and one-half plus one and one-half equals three." Rather expect them to say something like, "There are 3 squares altogether—2 whole squares and one folded over."

NUMBER LINE CHUNKS



A large copy of such a number line chart as this can be put to good use in the classroom. It can be used to demonstrate that it is possible to jump from 2 to 12, or 5 to 15, or 7 to 37 with great ease. Pupils won't need to count to do this, either. Every jump down ↓ is equivalent to adding 10. A jump down ↓ and $\overset{\frown}{2}$ is equivalent to adding 12. A jump down ↓ and $\overset{\frown}{2}$ is equivalent to adding 8. To find the sum of 10 and 17, you start at 10, jump ↓ one row and $\overset{\frown}{7}$ and you're on 27. Or, start at 10, jump $\overset{\frown}{7}$, and then jump ↓ one row.

Help your pupils learn how to move about this number line grid. They will enjoy exercises like these: "Where will you land if you start at 0 and take three jumps down? Four jumps down? Two jumps down? Six jumps down? Ten jumps down?

"What are some ways in which we can make the trip from 11 to 22? From 5 to 12? From 3 to 24? From 7 to 18? From 25 to 37? From 111 to 123?

"Where are we if we start on 15, jump down two rows and four jumps to the right? (15 + 20 + 4 = 39.) If we start at 15, take four jumps to the right, and jump down two rows? (15 + 4 + 20 = 39.) If we take two jumps to the right, jump down two rows, and jump two to the right? (15 + 2 + 20 + 2 = 39.)"

Have pupils jump forward or back, and then up, to subtract on the number line. "Just as there are several ways to get from 15 to 39, there are also several ways to get from 39 to 15."

When pupils understand how to use the number line chart, it is possible to play games with it. For instance, one game is started by saying, "I'm thinking of a number. Start at 15, jump three rows down and four to the left. What is the number?" Whoever can tell you first what the number is becomes the leader and has to think of a number for which he must give appropriate clues. Some pupils may become so proficient that they won't need to use the number line chart. Encourage such pupils to "do it in their heads."

Other games could proceed like this: "Start at 0, add 11. Add 11 more. Add 11 more. Where are you?" Or, "Start at 3, add 6. Add 3 more. Add 6 more. Where are you?" Use repeated additions and subtractions of 3, 4, 5, 6, 7, etc., and ask pupils to look for patterns as they progress.

The structure of the decimal system is emphasized in the manner the number line chart is designed. Bases for understanding the algorithms of adding and subtracting can be firmly established by developing the ability to jump freely on the chart. Enterprising children will discover many shortcuts for making jumps. The search for shortcuts should be encouraged, because in discovering shortcuts pupils also discover basic patterns in the decimal system of numeration.

THE GAME OF TENS AND ONES

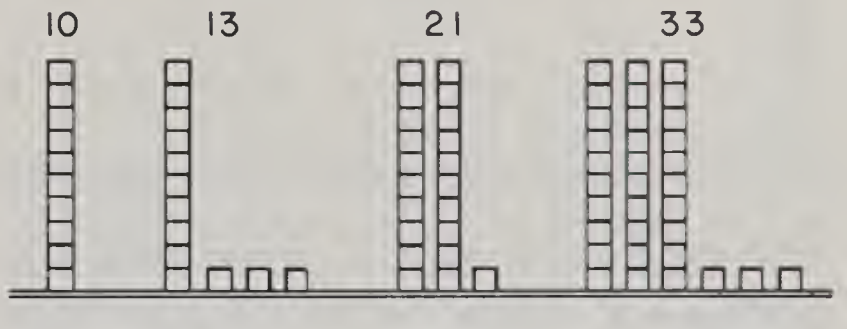
Consider a set of 9 cards, each containing 10 unit squares (about 2 inches by 2 inches or 2½ inches by 2½ inches):

and 9 cards, each a unit square in size:

This set of cards has an interesting property which points up the decimal system of numeration. Any number of unit squares from 1 to 99 can be represented by a set of such cards in only one way. For example:

8 . . . eight small cards
10 . . . one large card
13 . . . one large and three small cards
42 . . . four large and two small cards

A chalkrail is a handy support for arrangements of the cards. Such a method of display provides a handy way of recording the results of counting or of other means of finding the number of unit squares:



After a series of such results are noted, take the cards away. The children are then asked to place cards back on the rail so there are an appropriate number of unit squares in each group.

Another activity can begin with several numbers written on the board—with the children asked to find an appropriate combination of large and small cards for each number:

20 14 30 13 22

(Notice that these numbers can be selected, as in this case, so that every card in the set of 18 is required . . . since 20 + 14 + 30 + 13 + 22 = 99. Digits occurring in the tens-position are 2, 1, 3, 1, 2

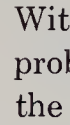
—and the sum of these is 9; digits in the units-position are 0, 4, 0, 3, 2—and the sum of these digits is 9 . . . and we have 9 of each size of card.)

Numbers can be listed by giving indicated sums:

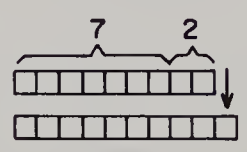
$$\begin{array}{r} 2 \\ +\,1 \\ \hline \end{array} \qquad 4 + 3 \qquad 10 + 10 \qquad \begin{array}{r} 20 \\ +\,10 \\ \hline \end{array}$$

Next, consider these indicated sums one at a time, because the very limited supply of cards presents a bit of a problem:

7 + 3 = ___ 6 + 4 = ___ 4 + 6 = ___



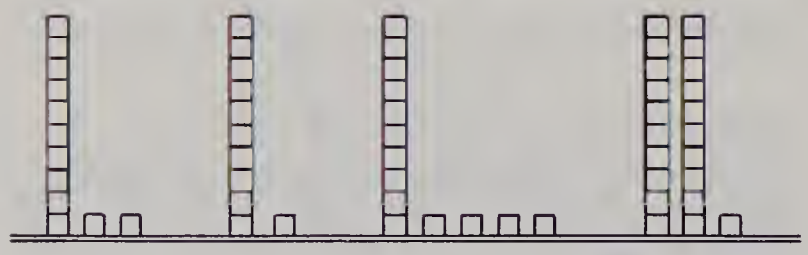
With only 9 unit pieces, 7 + 3 pieces presents a problem. There is only one way out—to use one of the larger pieces. Consider 7 + 2—and compare this with:



So, 7 + 3 would need more than 7 + 2 unit pieces —and a large piece will fit this requirement.

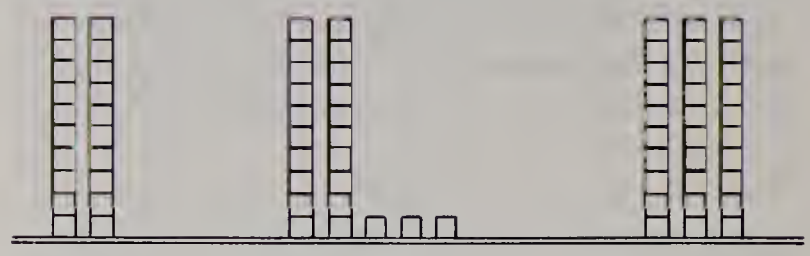
After much similar work, more difficult problems can be presented:

4 + 8 = ___ 9 + 2 = ___ 10 + 4 = ___ 20 + 1 = ___



and, eventually:

13 + 7 = ___ 15 + 8 = ___ 15 + 15 = ___



In each case, the idea of "carrying" is anticipated. However, children are involved in manipulating and counting objects rather than being confronted just with numbers. Selecting an arrangement of cards with 15 + 5 units presents a problem—and its solution comes only with the realization that two large pieces are required (and, perhaps, noting that there is no other way to fulfill the requirements).

Each of these activities can be carried out with a set of 9 each of the white and orange Cuisenaire® rods, with 9 of the 1-unit and 9 of the 10-unit Stern blocks, and with any similar set of other representational materials. The important idea is that there be only 9 of the unit representations and only 9 of the 10-unit representations.

THE GAME OF 8'S, 4'S, 2'S, AND 1'S

We have 4 strips of cardboard ruled in unit squares to suggest the idea of 1, 2, 4, and 8:

How many numbers can be represented by these strips, singly or in combination?

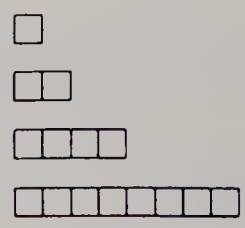
How many can we represent by using them 1 at a time?

How many by using 2 strips at a time?

How many by using 3 strips at a time?

How many by using 4 strips at a time?

Keep a record of these results, including the appropriate mathematical sentences:

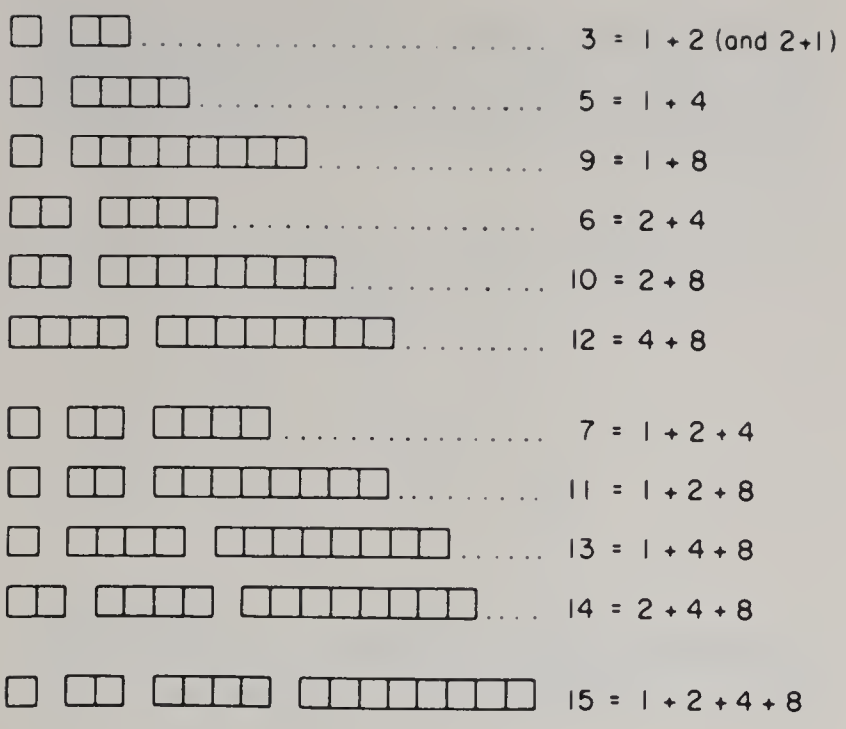


1 = 1

2 = 2

4 = 4

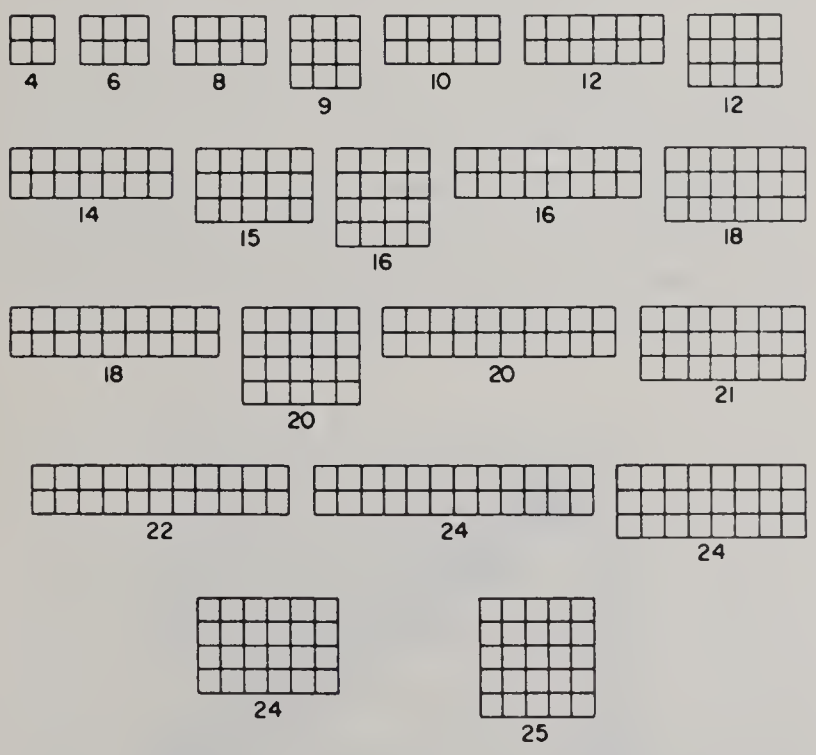
8 = 8



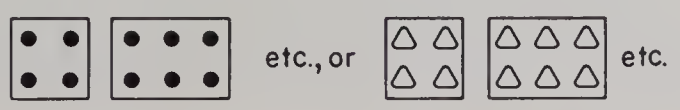
Does this list show representations for all the numbers 1 through 15?

PRIME NUMBERS

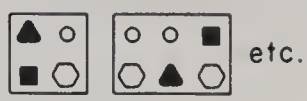
Make rectangular cardboard pieces marked with squares in rows and columns—with at least 2 squares in each row and in each column. The backs of these pieces are blank:



(You may prefer other designs, such as:



or use different kinds of things to count, such as:



The shapes are immaterial—we are only interested in *things to count* arranged in rows and columns.)

If represents 4, what numbers do the other pieces represent?

Are there any numbers that are suggested by more than 1 of the pieces? Yes—12, 16, 18, 20, 24. (24 is suggested by 3 of the pieces.)

Can any other numbers less than 26 be suggested by similar cardboard pieces, with the objects arranged with at least 2 in each row and column? No. But we must agree that all columns and rows must be full, thus excluding a nonrectangular piece such as:

Which numbers greater than 1 have we missed? . . . (2, 3, 5, 7, 11, 13, 17, 19, 23.) Such numbers are called *prime numbers*. Some children might like to find some prime numbers that are larger than 25.

Can any numbers less than 26 be represented by such pieces in a way not shown? (No—assuming that these pieces can be turned in any position—

can be turned to look like this .)

Some teachers may want to have the class make its own sets of cardboard pieces.

Pairs of Pieces

What numbers can be represented by pairs of these pieces? Keep track of the combinations in appropriate mathematical sentences. Display each combination, setting them on the chalkrail or in any other place where the class can see them.

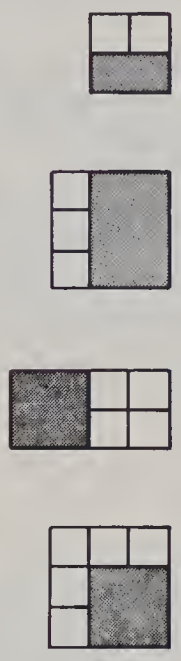
$4 + 6 = 10$ and $6 + 4 = 10$
$4 + 8 = 12$ and $8 + 4 = 12$
$4 + 9 = 13$ and $9 + 4 = 13$
$15 + 4 = 19$ and $4 + 15 = 19$
$15 + 6 = 21$ and $6 + 15 = 21$
etc.

The list becomes quite long.

If one piece of the manipulative material used in

the previous activity is flipped over and placed on top of another, and we count the squares we can see, how many numbers less than 25 can be suggested? (The flipped-over piece must fit within the other piece— cannot be placed on top of).

Keep a record with the appropriate mathematical sentences.

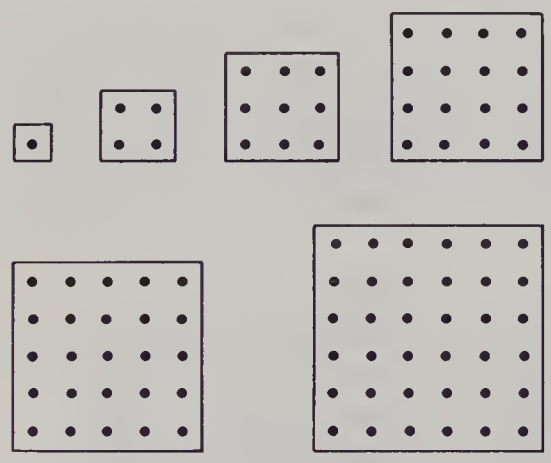


12 − 6 = 6
16 − 9 = 7
12 − 4 = 8
18 − 9 = 9
25 − 15 = 10
20 − 9 = 11
16 − 4 = 12
24 − 11 = 13
18 − 4 = 14
25 − 10 = 15
25 − 9 = 16
25 − 8 = 17
24 − 6 = 18
25 − 6 = 19
24 − 4 = 20
25 − 4 = 21

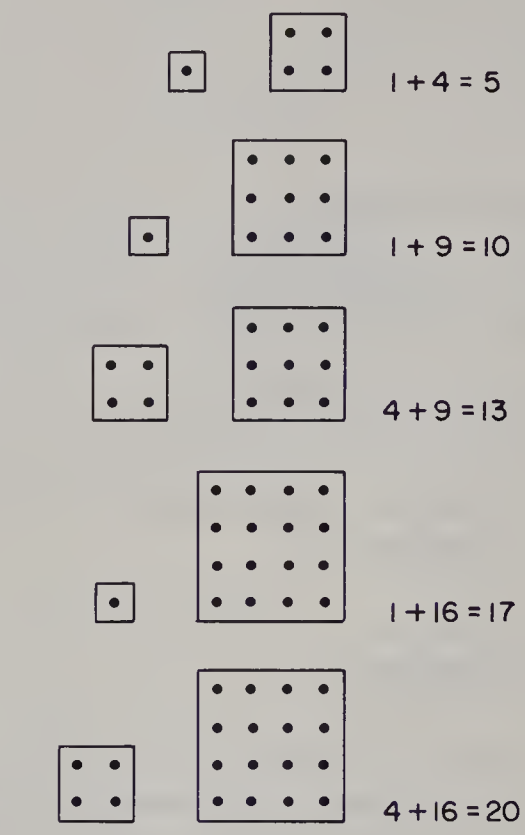
All numbers from 2 through 21 can be suggested by this means, and many can be suggested in several different ways.

SQUARE NUMBERS

Here are cardboard pieces with dots arranged in rows and columns, with the same numbers of dots



in the rows as in the columns. (Again, use any similar design you prefer.) The backs are blank. If these pieces represent the numbers 1, 4, 9, 16, 25, and 36, how many numbers can be suggested by a pair of pieces placed side by side? Keep a record with the appropriate mathematical sentences:



A Helpful Kind of Record

You might introduce and discuss another way to keep a record that condenses the results and helps us to be sure we didn't miss any:

+	1	4	9	16	25	36
1		5	10	17	26	37
4			13	20	29	40
9				25	34	45
16					41	52
25						61
36						

If the cells above the shaded diagonal record the results:

$1 + 4 = 5$
$1 + 9 = 10$
$4 + 9 = 13$
etc.

then the cells below the shaded diagonal can be used to record the results:

$4 + 1 = 5$
$9 + 1 = 10$
$9 + 4 = 13$
etc.

An Extension

How many other numbers can be represented by using the pieces three at a time? By using the pieces four at a time?

$1 + 4 + 9 = 14$
$1 + 4 + 16 = 21$
$4 + 16 + 36 = 56$
$9 + 16 + 25 = 50$
etc.

$1 + 4 + 9 + 25 = 39$
$1 + 4 + 9 + 36 = 50$
$9 + 16 + 25 + 36 = 86$
$4 + 9 + 16 + 25 = 54$
etc.

Notice that the number 50 is included in both of the last two lists. How many other numbers can be represented *in more than one way* by combinations of these cards?

$9 + 16 + 25 = 50$
$1 + 4 + 25 = 30$
$25 + 4 = 29$
etc.

$1 + 4 + 9 + 36 = 50$
$1 + 4 + 9 + 16 = 30$
$4 + 9 + 16 = 29$
etc.

Using the same manipulative materials as in the previous activity, other numbers can be represented by flipping over and placing one of the pieces on top of one of the others. What numbers can be represented by the dots you can see?

9 − 1 = 8

16 − 9 = 7

$9 - 4 = 5$
$4 - 1 = 3$
$16 - 1 = 15$
$25 - 4 = 21$
etc.

Again the results could be organized in a chart similar to the one used earlier.

−	1	4	9	16	25	36
1	0					
4	3	0				
9	8	5	0			
16	15	12	7	0		
25	24	21	16	9	0	
36	35	32	27	20	11	0

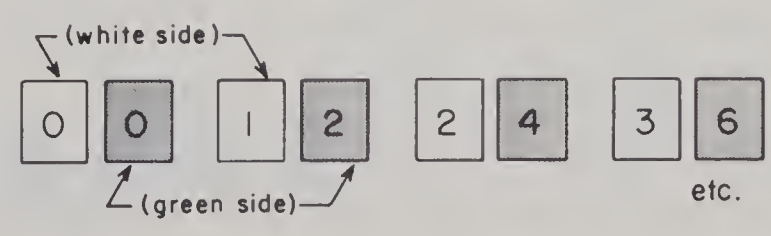
Arranging the results in this way indicates interesting patterns that the pupils might not have seen otherwise.

SURPRISES WITH TWO-SIDED NUMBER CARDS

Here is a simple set of cards that will provide needed practice in basic combinations and probe further into fundamental mathematical ideas.

Activity 1

The group of cards could be, say, white on one side and green on the other. The white sides are labeled 0, 1, 2, 3, etc., and the green sides are labeled 0, 2, 4, 6, etc.—the numbers indicated on the green side are twice those on the white side:

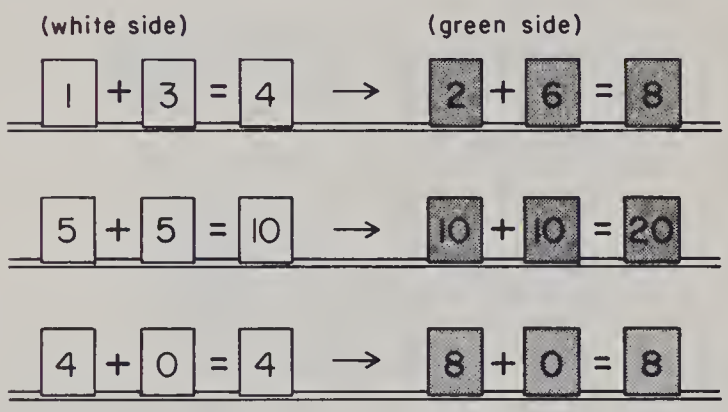


(It is helpful to have at least two copies of each card—preferably three.)

Use the white cards spaced along the chalkrail and write appropriate signs on the board to make a true statement, such as:

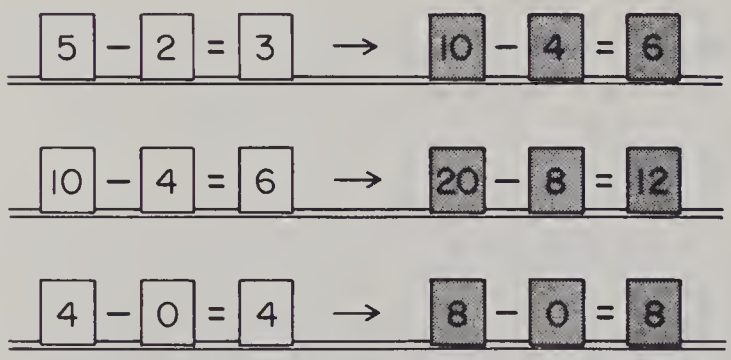
Now turn them over and consider the resulting statement:

It's a true statement, also. Try other examples:



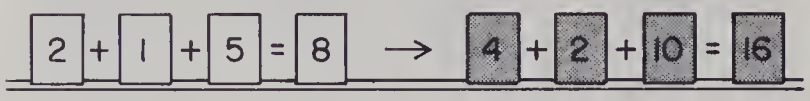
In each case we have the same outcome—a true statement using the numbers on the white side leads to a true statement when the cards are turned over.

Try sentences involving subtraction:

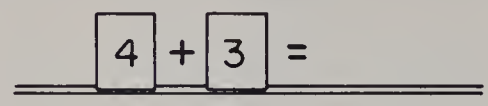


Clearly these surprising results are not accidental.

Try more complex sentences, such as:



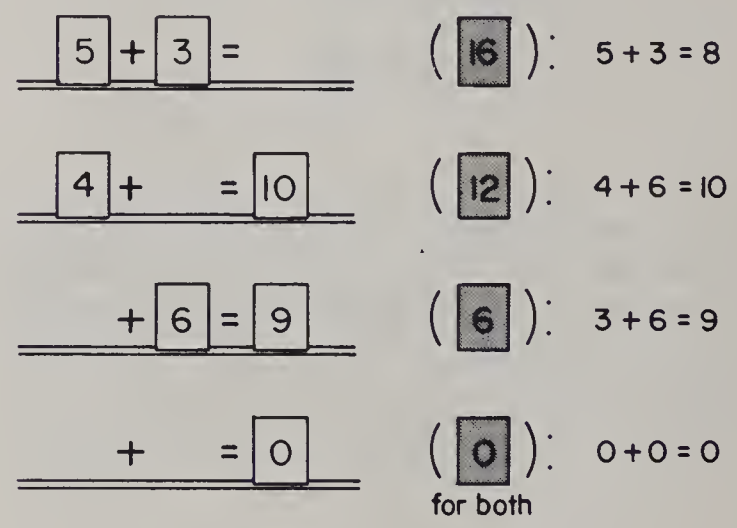
Suppose that we wish to complete a sentence to make it a true statement, using this set of cards. We see:



and we need to select a card from the set displayed

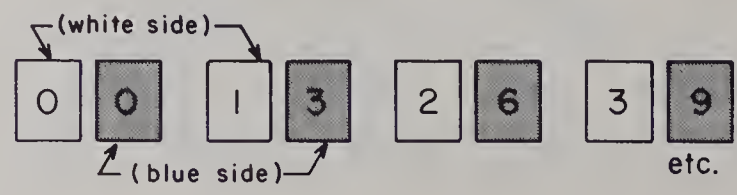
in such a way that we can see only the green sides. Which one do we need? Clearly the one with 14 on the green side, because the white side will be labeled 7, and 4 + 3 = 7.

Which cards shall we select from the set of green cards so they will make the following into true statements when the white side is used?

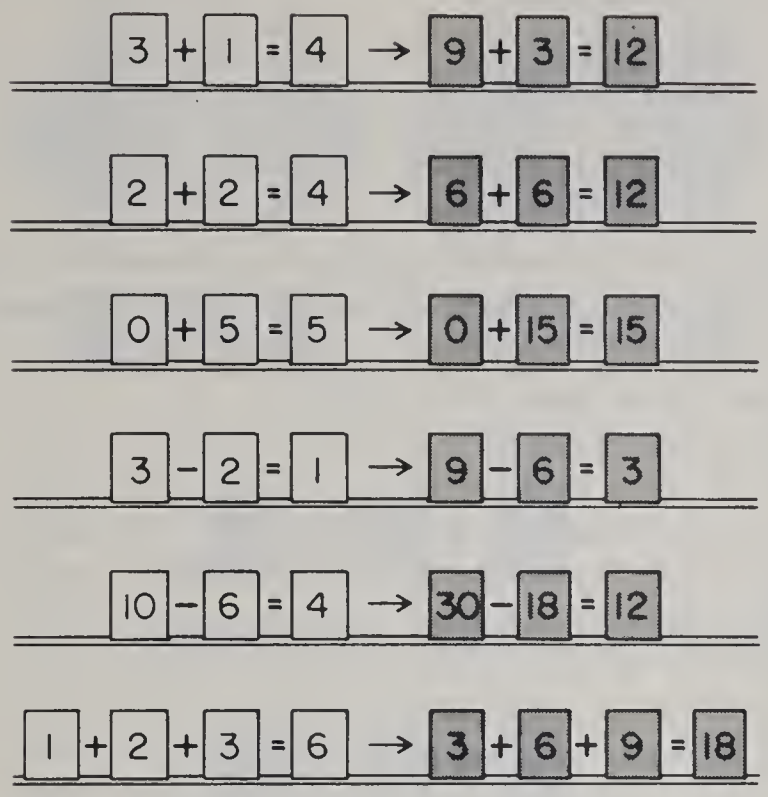


Activity 2

Try the same kind of activity with another set of cards—one side blue, perhaps, labeled with a number three times the number written on the other side.

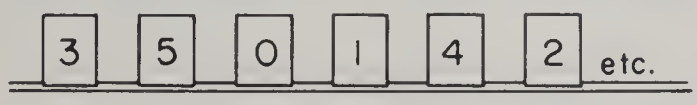


Here are examples of results:



Line the cards up on the chalkrail with the white side showing, and in order:

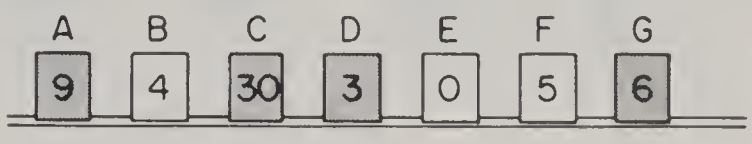
"What number is listed on the back of 2?" Regardless of the response, turn the card around—to see for sure that it is 6. Try other cards until some familiarity develops. Then arrange the cards in a random order:



Next, try the same activity with the blue sides showing and, at first, in increasing order:

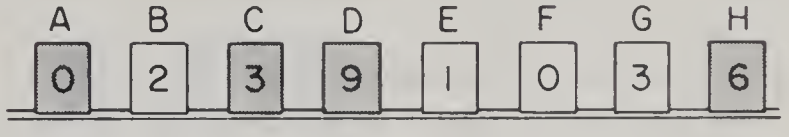
"What is on the other side of the card whose blue side is labeled 9?" . . . and check to be sure the answer is 3. Etc. Next, try the cards in a random order:

To challenge the faster children, combine the two activities by placing the cards in random order, some with the blue side showing and some with the white side showing. Letters written on the chalkboard may help specify the card to be discussed:



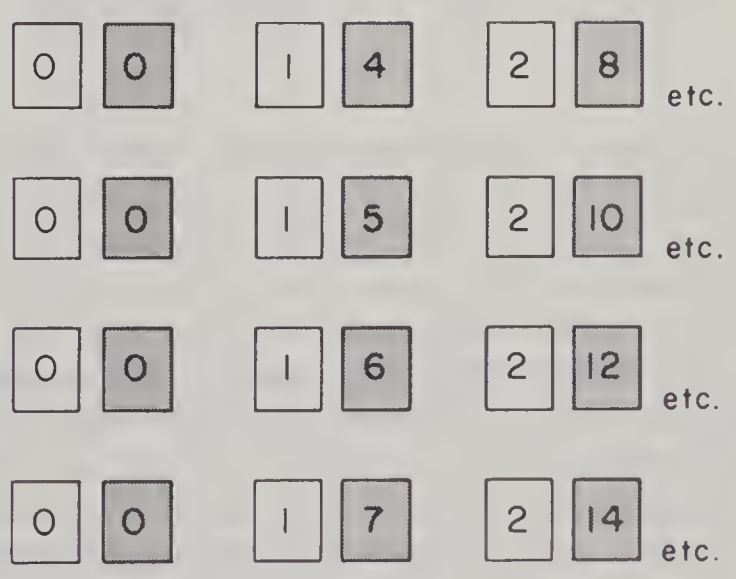
"What's written on the back (or white side) of A? . . . (or, the card labeled 9 on the blue side?) "Describe the card under the letter D." (3 is written on the blue side and 1 is written on the plain or white side.) "Which card has 10 written on the white side?"

If two or more of each card are available, they can be arranged in random order with some showing the blue sides and some showing the white sides. Again, letters on the chalkboard would help specify the cards without physical pointing:

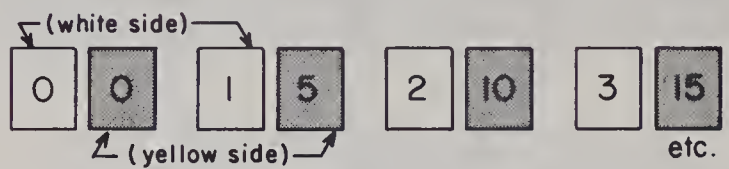


"Which cards are alike? . . . Which card is like the one under the letter H?" Answer: The one under the letter B. Etc.

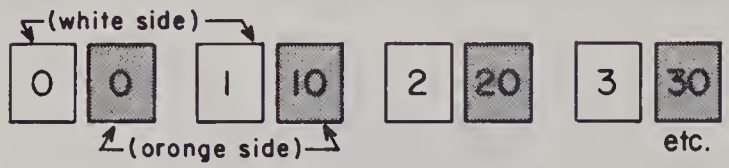
Additional sets of cards can be made using multiples of 4, 5, 6, 7, 8, 9, 10, etc:



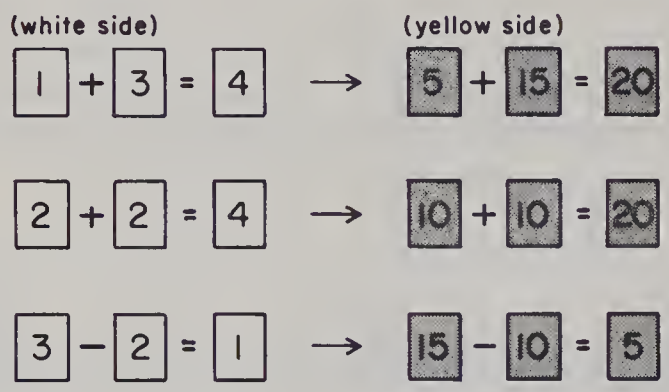
Two of the sets of cards that deserve special attention are those for multiples of 5 and multiples of 10:

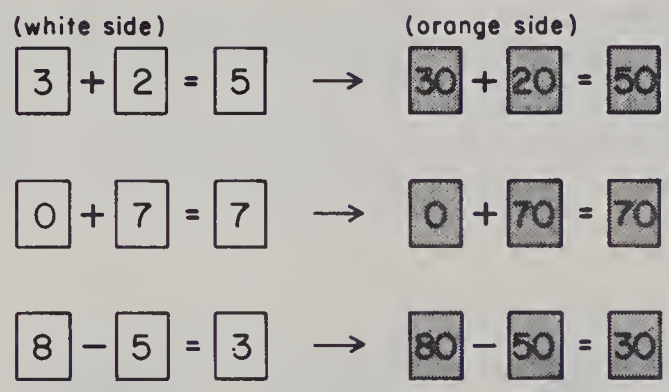


and:



Some of the results that follow are:

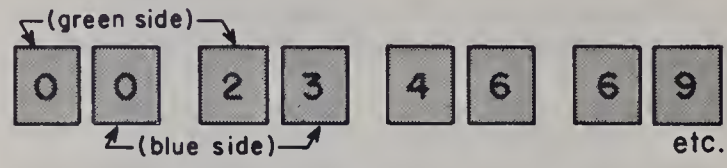




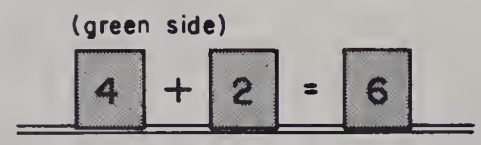
Each of the activities suggested earlier can be developed around any of these sets of cards by posing questions that require both memory of fundamental combinations and problem solving.

An extended use of these cards (or new sets of cards) provides additional levels of complexity and practice for all of the children or, at least, the more capable.

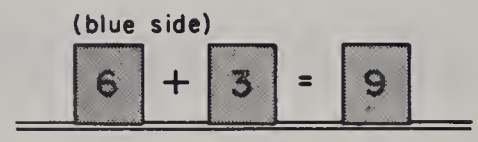
Consider a set of cards made by clipping one of the sets to another—with white sides together and only the colored sides showing. For example, the multiples of 2 and the multiples of 3 are clipped together (or a new set used), so that the sides showing are:



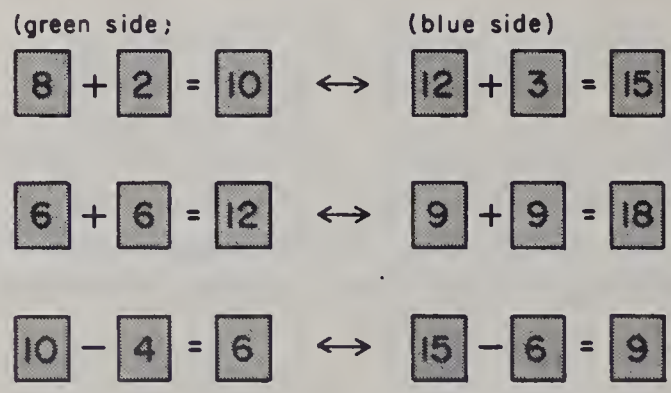
Make up true sentences using the green sides. Can you anticipate what will happen when the cards are turned over? Will the resulting statement be true in each case? For example:



Turn the cards over and find the results:



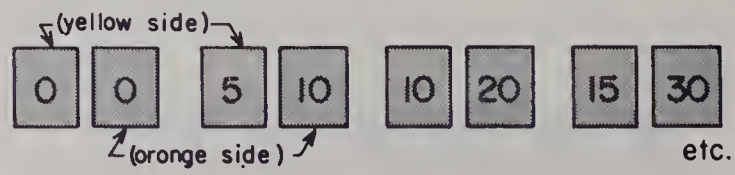
Make up other true statements using cards with sides of the same color showing. Turn the cards



over. Will the resulting sentences be true statements?

Now each of the activities outlined above can be approached again with this new set of cards.

Any two sets of cards can be used to create challenging problems for children—problems that require both memory and active thought. Here are a few examples showing multiples of 5 (shown as yellow) and multiples of 10 (shown as orange). The combinations begin with:



and sample sentences might be:

(yellow side) (oronge side)

5 + 10 = 15 → 10 + 20 = 30

15 + 15 = 30 → 30 + 30 = 60

25 − 15 = 10 → 50 − 30 = 20

Some aspects of the foregoing suggest activities that are definitely of the enrichment type. As the pupils become familiar with the cards, their responses will be a conclusive guide as to the extent any activity ought to be pursued. The more difficult developments also serve to indicate that this activity will be used at advanced levels since fundamental mathematical relationships are being explored. As the children grow in understanding, new dimensions will be added. Consider just one future extension to get the depths that will be studied.

A set of cards are labeled as follows:

Now suppose that the green cards are used to form addition statements. When they are turned over, the statement is changed to one about multiplication. For example:

(green side) (blue side)

1 + 3 = 4 becomes 2 × 8 = 16

3 + 0 = 3 becomes 8 × 1 = 8

2 + 2 = 4 becomes 4 × 4 = 16

In other examples, sentences about subtraction with green sides become sentences about division with blue sides:

(green side) (blue side)

4 − 3 = 1 becomes 16 ÷ 8 = 2

2 − 0 = 2 becomes 4 ÷ 1 = 4

This example indicates that a basic idea initiated at an early level can be used as a foundation for an extension of that idea.

FREQUENCY RECORDS

The idea of *frequency* is significant. Common notions such as those expressed by phrases like *most often, more often than, not at all, only once,* etc., will grow eventually into the ideas involved in probability, statistics, and even cryptography.

A Class Project

Suppose that we decide on a class project to record the frequency with which each of the 26 letters of the alphabet occurs on pages 10 through 14 of some book. An organization job presents itself. No one ought to tackle the job by himself. The final goal, in terms of a summary, helps determine the organization. Each letter of the alphabet can be written across the board. A blank grid makes the tasks ahead more specific:

	A a	B b	C c	D d	E e	F f	G g	etc.
Page 10								
Page 11								
Page 12								
Page 13								
Page 14								
Totals								

Five teams might tackle a page each. Or, several teams might assume responsibility for specific letters of the alphabet. In either case, each team must have an appropriate grid for keeping its records. The exact nature of these, of course, will depend on the organization selected.

Results are not very interesting, nor are they

very significant, until they are compiled and totaled. Surprising outcomes soon become apparent.

Some letters are used very often, others very seldom. "Suppose that we try this again, perhaps with another book this time. Do you think the same kind of results will occur? Let's keep this record so we can have it for comparison when we do this job again. Which letter was used most often? How many are not used at all? Which five letters were used most often? How many letters were used more often than M?" Etc.

Everyone in the class has contributed to the creation of the final picture. This is a group accomplishment. It got results worth talking about; results that will be used again in the future as a basis for comparison. (And it should be repeated at least once and the results compared.)

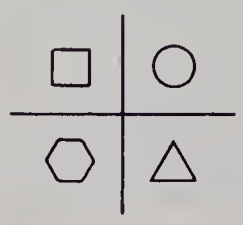
You may have a group that would like to find out the frequency of one-letter, two-letter, three-letter words on those same or other pages.

This is not busywork. There is a goal to be reached. It produces subject matter for discussion. It provides a basis for comparison. The ideas of *frequency* and *predictability* will take root, to be nourished by similar activities such as: "How many days does it rain or snow? How many children are absent each day during the week or month?"

AND-OR GAMES

The distinction between the meanings of the words *and* and *or* as they are used in mathematics is of fundamental importance. While mastery of the distinction by pupils is not our concern at the present, it is necessary that you as a teacher become aware of it and begin to ask questions such as those below. The distinction, of course, is purely a matter of definition—explaining an agreement of mathematicians as to the use of these words.

Without too much discussion, let's try some illustrations:

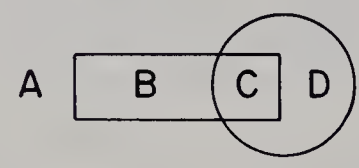


"How many shapes are above the horizontal line? . . . and what are they?" (Two, the square and the circle.)

"How many shapes to the right of the vertical line? . . . and what are they?" (Two, the circle and the triangle.)

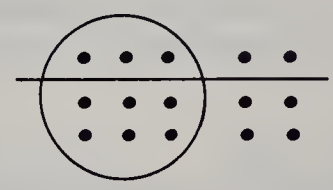
"What shapes are both above the horizontal line *and* to the right of the vertical line?" (Only the circle—it is the only shape that is both above *and* to the right.)

"What shapes are either above the horizontal line *or* to the right of the vertical line?" (The square, the circle, and the triangle.)



"What letter or letters are both inside the circle *and* inside the rectangle?" (Only C.)

"What letter or letters are inside either the rectangle *or* inside the circle?" (B, C, and D.)



"How many dots are both inside the circle *and* above the line?" (3.)

"How many dots are both inside the circle *and* below the line?" (6.)

"How many dots are both outside the circle *and* above the line?" (2.)

"How many dots are both outside the circle *and* below the line?" (4.)

Ask these same questions, using "either . . . or . . ." instead of "both . . . and . . ."

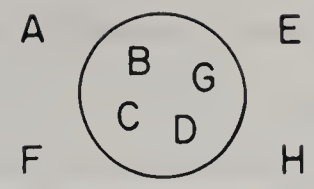
"How many dots are either inside the circle *or* above the line?" (11.)

"How many dots are either inside the circle *or* below the line?" (13.)

"How many dots are either outside the circle *or* above the line?" (9.)

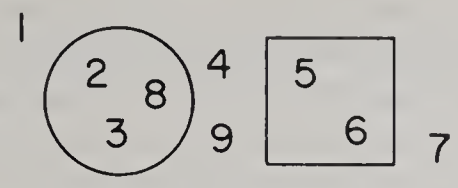
"How many dots are either outside the circle *or* below the line?" (12.)

Pupils should also have some understanding of the term *region*. Explain to the class that a simple closed curve of any kind establishes two *regions*: the *outside* and the *inside*. To demonstrate this, draw the following diagram on the board:



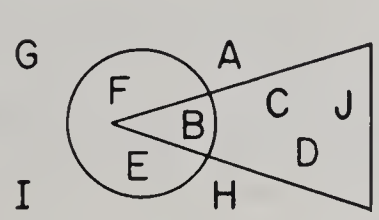
"Which letters are in the region inside the circle? Which letters are in the region outside the circle?"

Next, illustrate that the number of regions to be considered increases if more than one figure is involved. Draw the following diagram of a circle and a square on the board:



"How many numbers are in the region outside the circle?" (The numbers 5 and 6 are inside the square, but they are also outside the circle. The answer, therefore, is five.) "How many numbers are inside the square?" (Two.) "How many numbers are either inside the square or inside the circle?" (Five.) "How many numbers are both inside the square and inside the circle?" (Zero.)

Overlapping shapes make it especially necessary to be precise about what region is being discussed. Draw the diagram:



"How many letters are inside the circle?" (Three.) "How many letters are outside the circle?" (Seven.) "How many letters are inside both the circle and the triangle?" (One, B.) "How many letters are not inside both the circle and the triangle?" (Nine —all except B.) "How many letters are inside either the circle or the triangle?" (Six.) "How many letters are outside both the triangle and circle?" (Four.) "How many letters are inside the triangle but not inside the circle?" (Three.)

SQUARED PAPER PATTERNS

Squared paper invites mathematical activity. It is never too early to begin; few persons will ever grow beyond graph paper arithmetic—in mathematics, in engineering, in business, in reading the newspaper. No more effective method has been discovered than work with graph paper to help reveal the simplest to the subtlest relationships.

Begin with the lowly *stair step.* Color a square with any crayon. Then color the one under it a different color and another one still a different color, etc. (Cuisenaire® rods or Stern structural blocks can serve as models to be copied on squared paper.)

W						
R	R					
G	G	G				
P	P	P	P			
Y	Y	Y	Y	Y		
D	D	D	D	D	D	
K	K	K	K	K	K	K

W - white
R - red
G - green
P - purple
Y - yellow
D - dark green
K - black

Make another set of *stair steps.* Then, repeat the color combinations in an upside down set that fits on top:

W	P	P	P	P
R	R	G	G	G
G	G	G	R	R
P	P	P	P	W

And, back-to-back stairs:

			W	W			
		R	R	R	R		
	G	G	G	G	G	G	
P	P	P	P	P	P	P	P

A *nest of squares* is an interesting pattern. Shade one square one color. With another crayon, color all the squares around it. Build as big a nest as you wish.

	O	O	O	O	O	
	O	N	N	N	O	
	O	N	M	N	O	
	O	N	N	N	O	
	O	O	O	O	O	

Let a nest of squares grow out of a corner:

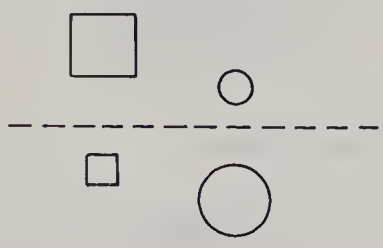
U	U	U	U	U
K	K	K	K	U
Y	Y	Y	K	U
G	G	Y	K	U
W	G	Y	K	U

This last pattern will help reveal the connection between *odd numbers* and *square numbers*. The pattern suggests the following table:

$$1 = 1$$
$$4 = 1 + 3$$
$$9 = 1 + 3 + 5$$
$$16 = 1 + 3 + 5 + 7$$
$$25 = 1 + 3 + 5 + 7 + 9$$

THE GAME OF FEWEST QUESTIONS

Here is a game that strengthens number concepts, expands vocabulary, and develops powers of reasoning. It is called *Fewest Questions*. You can start with a very simple example on the chalkboard or on a piece of oak tag:



Some preliminary questions might be: "How many shapes are there above the line? How many are there below the line? How many are there altogether? How many are squares? How many are circles? How many are large? How many are small?"

Now to the game. "I'm thinking of one of those shapes. Try to guess which one it is. We'll keep score of how many questions you ask." Select a child as scorekeeper at the board to make a tally each time the guesser asks a question. Except at the outset, discourage pointing to the diagram in asking questions.

"Is it the big square?"

"No, it's not the big square. That's your first question."

"Is it the big circle?"

"No, it's not the big circle. That's your second question."

"Is it the little circle?"

"No, it's not the little circle. That's your third question. Do you know the shape I'm thinking of?"

Of course, if the guesser remembers the questions, he will know that it has to be the "little square," but he may keep on asking until he gets the answer.

After a few such guessing sequences (in which you may take the liberty of saying *No* as often as you logically can), offer to be the *guesser* yourself.

"I'm going to turn my back. Barney will come to the board and point to one of the shapes. You will all know the one he selects—but I won't. Then I'm going to find out by asking just two questions!"

Your questions may be:

1) Is it a square?

2) Is it above the line?

There are only four possible combinations of answers. They are:

	A	B	C	D
Is it a square?	Y	Y	N	N
Is it above the line?	Y	N	Y	N

If the answers to both your questions are "Yes," it is A, the large square. If the answers are "Yes" and "No" in that order, it is B, the little square. If the answers are "No" and "Yes," it is C, the small circle. If the answers are both "No," it is D, the large circle.

You might ask another pair of questions:

1) Is it small?

2) Is it a circle?

Again, each combination of Yes-No answers is associated with one and only one shape. The secret is that each question must divide the group in half in a different way.

Now return to the children as *guessers* and *selectors*. (Some will get the idea right away. They've discovered the secret and they must keep it.) Change the diagram:

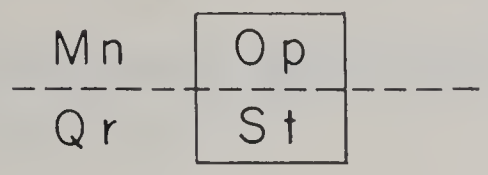
1) Is it a large letter?

2) Is it in the box?

Another pair of questions:

1) Is it outside the box?
2) Is it at the beginning or at the end (a or D)?

After the children become adept at asking the questions so they never need more than two, enlarge the diagram:



"How many large letters? How many small letters? How many letters altogether? How many letters above the line? How many letters below the line? How many inside the box? How many letters are both above the line and in the box?" (Only two—O and P.) "How many large letters are both below the line and not in the box?" (Only Q.)

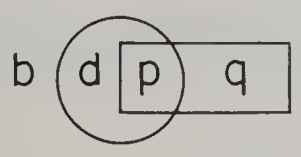
That last question suggests a group of three questions that will reveal any choice:

1) Is it a large letter?
2) Is it above the line?
3) Is it in the box?

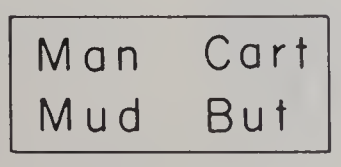
(1)	Y	N	Y	N	Y	N	Y	N
(2)	Y	Y	N	N	Y	Y	N	N
(3)	Y	Y	Y	Y	N	N	N	N
	O	p	S	t	M	n	Q	r

Variations in Designs

Fewest Questions is a favorite with children. Let them take their time in discovering the secret. You may vary the objects to include geometric shapes, letters of the alphabet, short words, numbers, etc. Below are a few variations in design with examples of the fewest questions that will expose any item chosen:

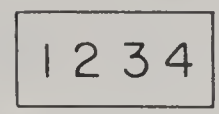


1) Is it in the rectangle?
2) Is it in the circle?



1) Does it begin with *M*?
2) Is it in the word?

1) Is it in the word *bark*?
2) Is it in the word *tub*?



1) Is it 1 or 4?
2) Is it more than 2?

CUISENAIRE® RODS

SECTION I: FREE-PLAY ACTIVITIES

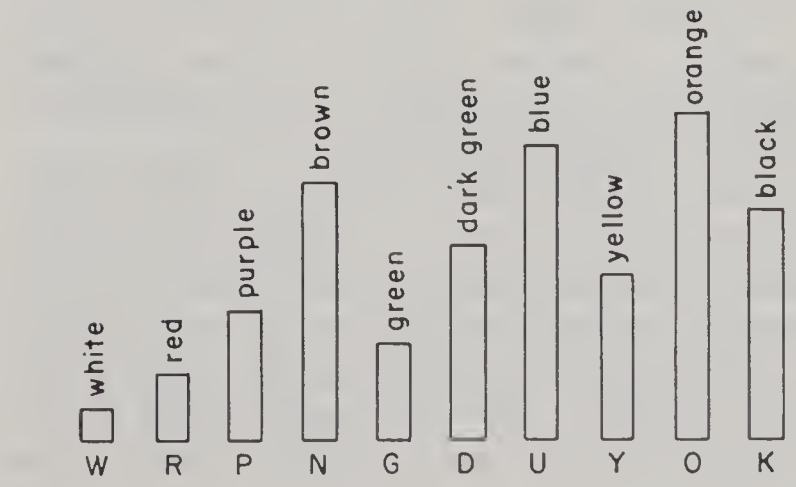
Teachers are in general agreement concerning the positive values of representational materials for learning. Where differences of opinion exist they usually center around answers to such questions as:

1. What kind(s) of materials are best for teachers to use?
2. What constitutes a balanced use of the materials selected?
3. What special teacher planning and classroom arrangements are best for maximum learning to occur by use of representational materials?

Your unique style of teaching will determine the kind of representational materials you select, the balance you establish among a variety of learning opportunities, and the plans and classroom arrangements you need for your pupils.

However, you may find certain observations and suggestions appropriate, whether you use commercially prepared structured materials like Dienes or Stern blocks, or teacher-constructed or pupil-constructed materials, many of which are described in this book.

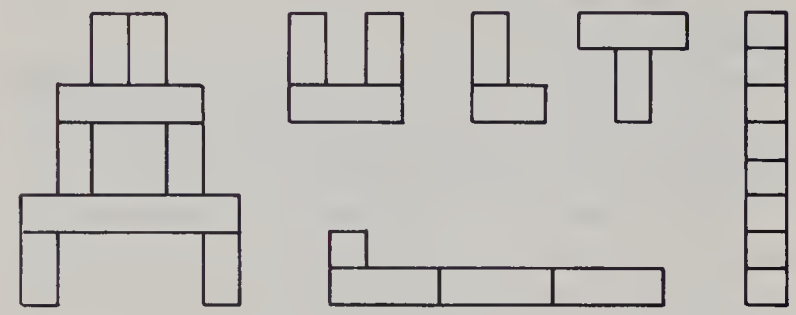
For purposes of communication, we shall rely on the colored wooden pieces known as Cuisenaire® rods.



At some point in the learning sequence you may plan to have your pupils explore representational materials. You may start your children off by displaying the materials on a table for them to examine and puzzle over. Or you may have provided sufficient materials so that each child will have his own set for independent investigation. In either case, you should plan ample periods of free play.

It is during free-play sessions that children meet their learning materials on their own terms. Their natural curiosity quickly involves them in conscious (and unconscious) actions upon the materials.

While manipulating Cuisenaire® rods, for example, children spontaneously construct, take apart, rearrange, and otherwise alter structures they build to represent an endless variety of houses, forests, bridges, boats, gardens, towers, trains, floors, letters of the alphabet, words, flowers, and so on.

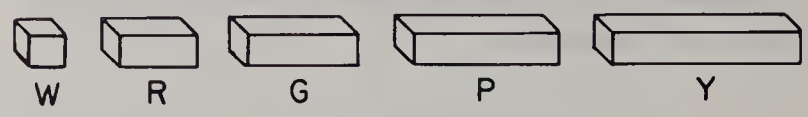


During the course of their play, children constantly make decisions concerning balance, symmetry, equivalence, and aesthetics. Most children are participating observers. They become quite involved in, and are frequently pleased with, their own artistry. Yet they don't hesitate to discard their own works to duplicate someone else's pattern if it suits their fancy.

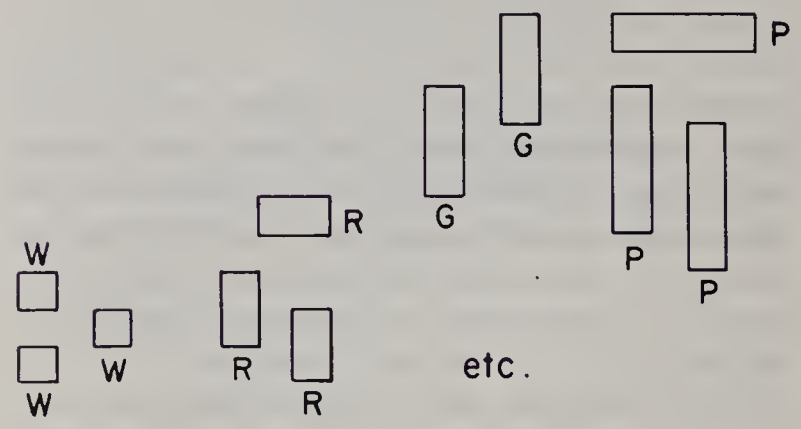
The benefits children derive from their free-play sessions can only be assessed in qualitative terms. Creativity, ingenuity, resourcefulness, willingness to explore alternatives, playing hunches, aesthetic satisfaction, and other highly personal experiences continue to defy quantification.

Children derive other significant benefits from their free-play sessions. As a consequence of their actions upon the materials, they acquire a fund of working knowledge of the properties and structure of their materials. A limited but fairly typical listing of children's discoveries would include the information pictured and stated below and on the next pages. Feel free to encourage and guide pupils in exploring these constructions. Rely on your common sense and intuition.

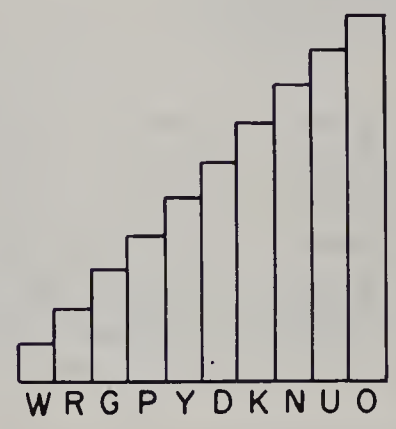
1. All Cuisenaire® rods are rectangular solids.



2. All rods of the same color are equal in length.

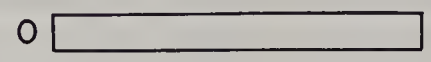


3. Ten different colors are used for the rods.

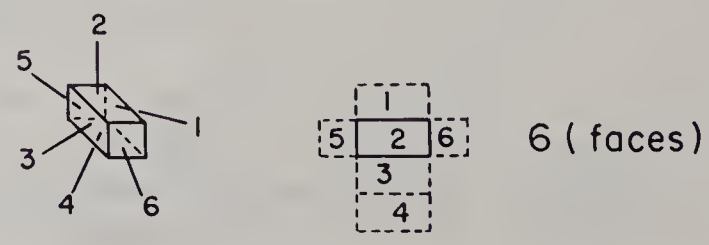


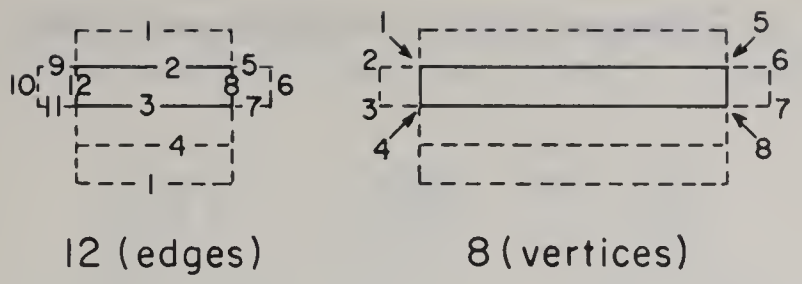
4. The white rods are the shortest (or smallest).

5. The orange rods are the longest (or largest).



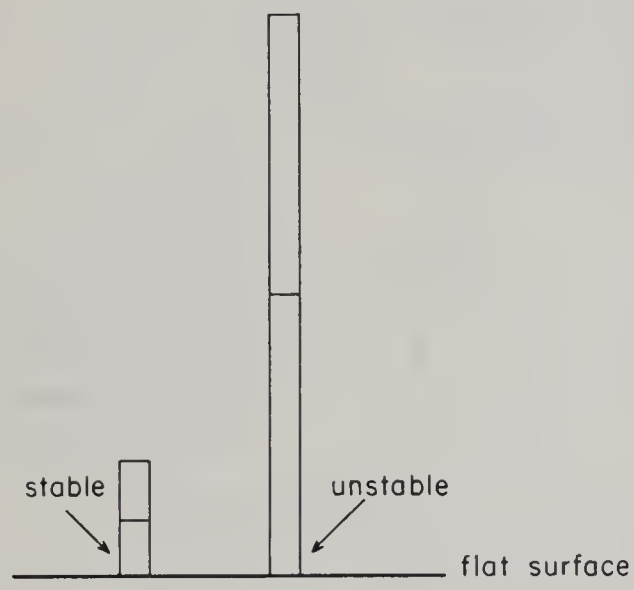
6. Each rod has the same number of faces (6), edges (12), and vertices (8) as every other rod.



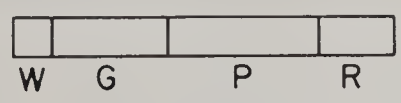


7. The white rods are cubes (each face is congruent to every other face).

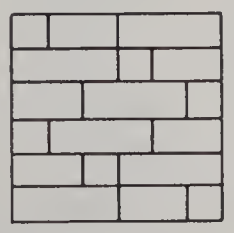
8. Vertical structures constructed with the larger rods are not as stable as those built with smaller rods.



9. Trains of rods can be constructed by placing them end to end.

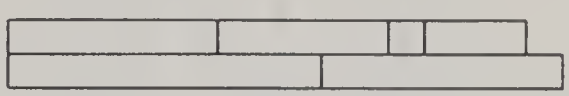


10. The positions of the rods in a train can be changed without changing the length of the train.

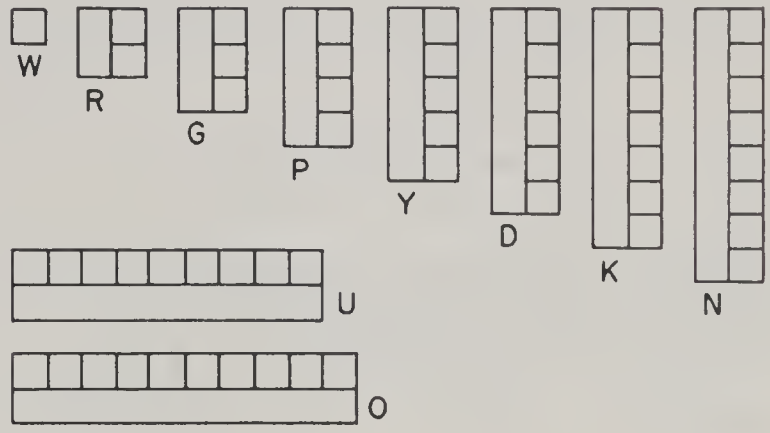


11. Some trains are equal in length to other trains.

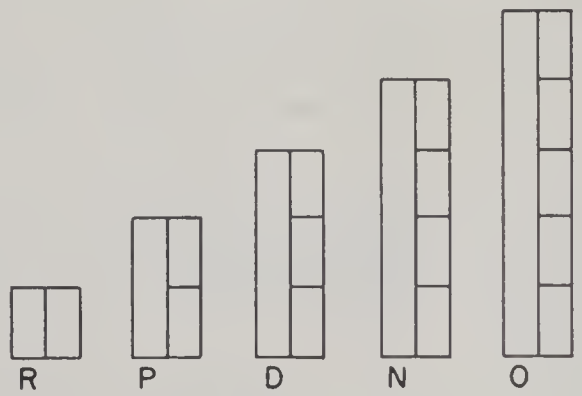
12. Some trains are not equal in length to other trains.



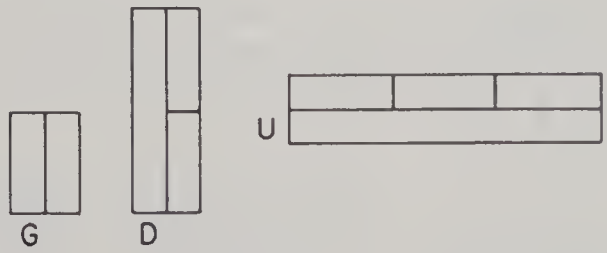
13. All rods are multiples of the white rod. Thus, a train of white rods can be constructed in such a way that its length is equal to that of any given rod.



14. Some rods are multiples of red rods. Thus, a train of red rods can be constructed with a length equal to that of any red, purple, dark green, brown, or orange rod.

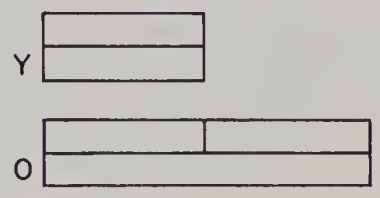


15. Some rods are multiples of light green rods (light green, dark green, and blue rods).

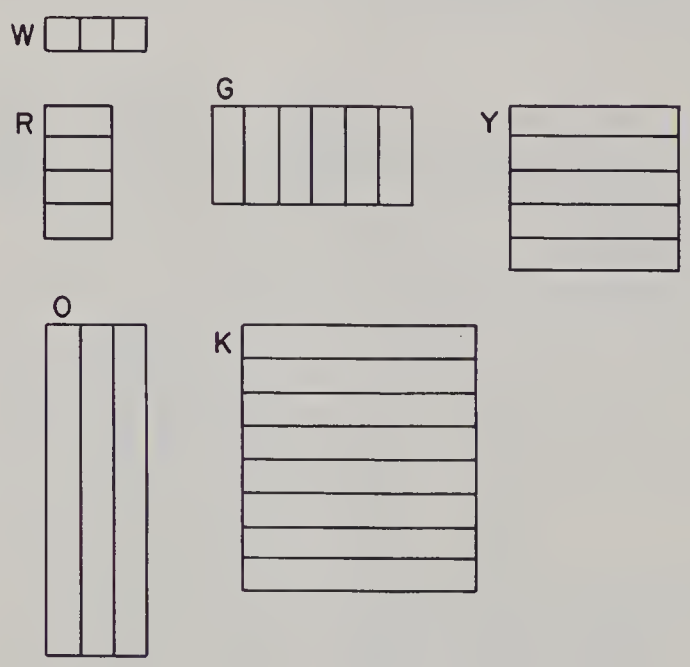


16. Some rods are multiples of purple rods (purple and brown rods).

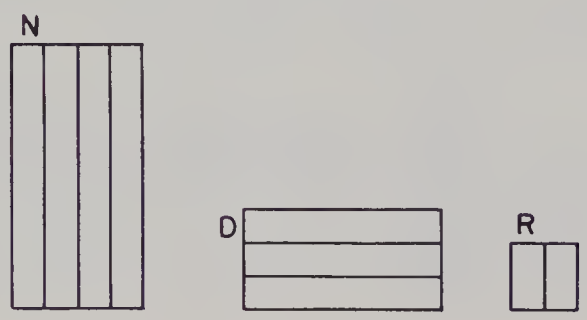
17. Some rods are multiples of yellow rods (yellow and orange rods).



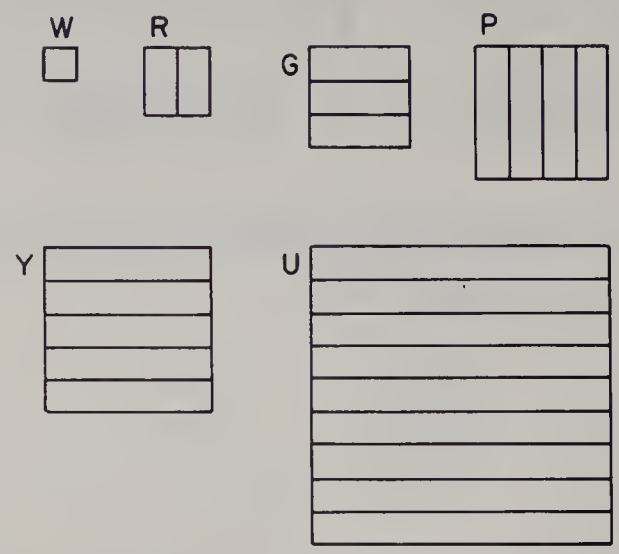
18. Floors of rods of the same color can be constructed by placing rods side by side.



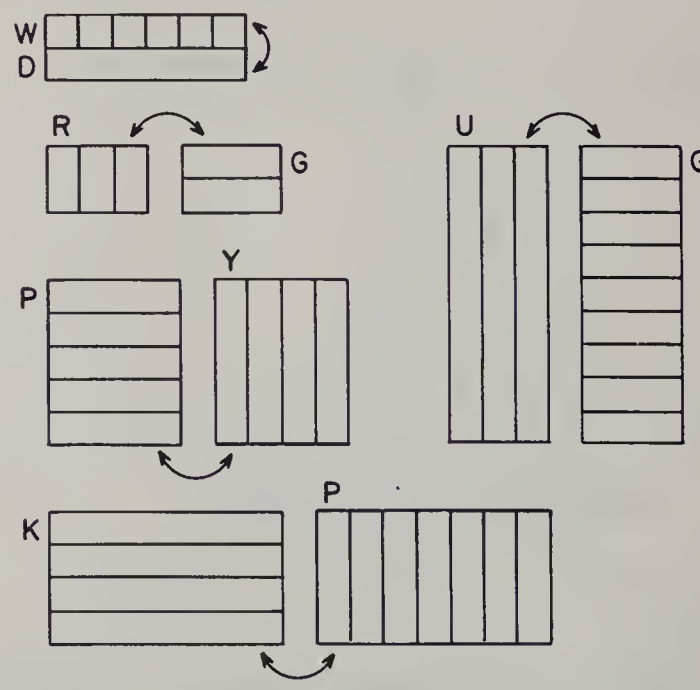
19. The upper face of some floors is a rectangular region.



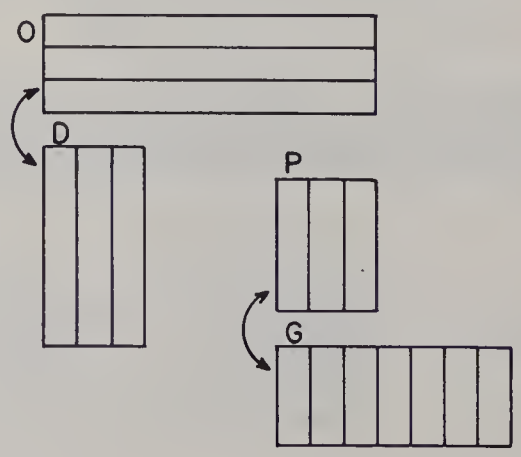
20. The upper face of some floors is a square region.



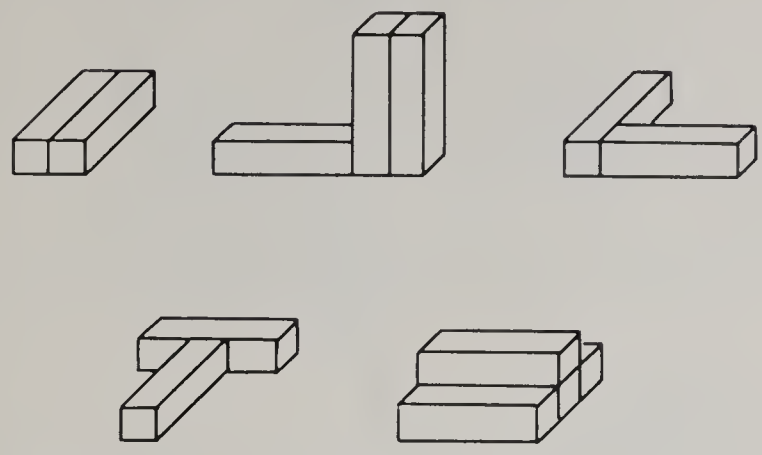
21. Some floors fit on each other.



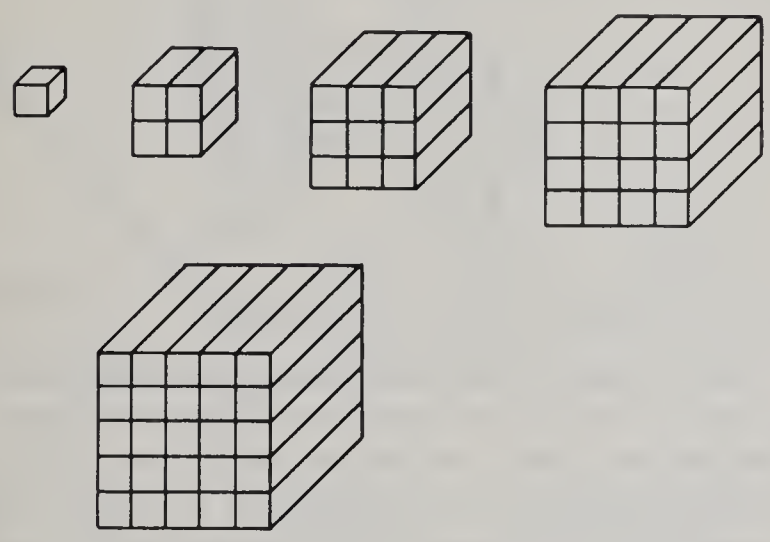
22. Some floors do not fit on each other.



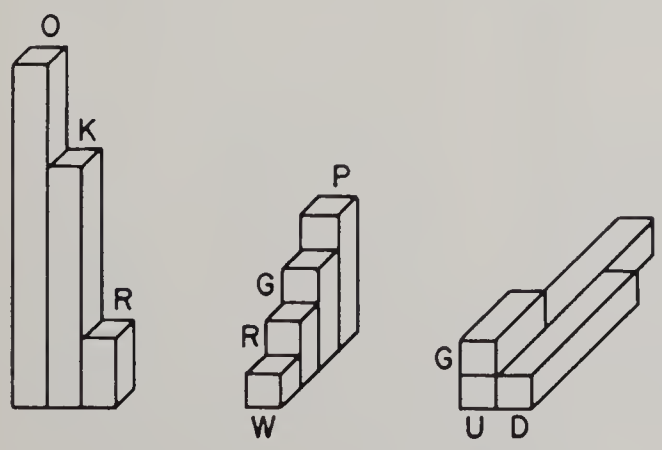
23. Rods of the same color can be used to construct a wide variety of rectangular solids.



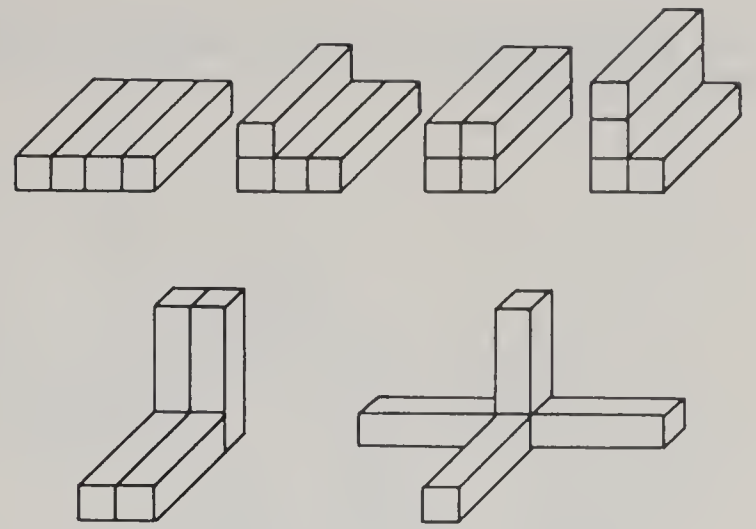
24. Some rectangular solids constructed are cubes.



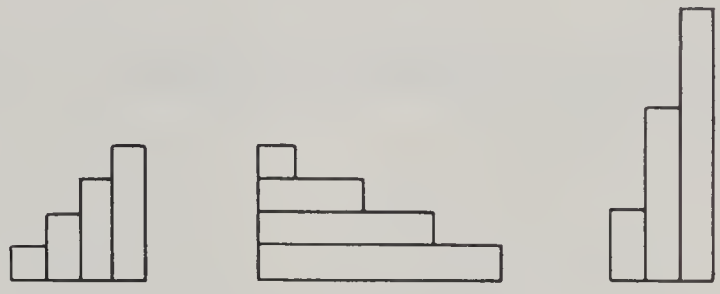
25. Rods of different colors can be used to construct irregular rectangular solids.



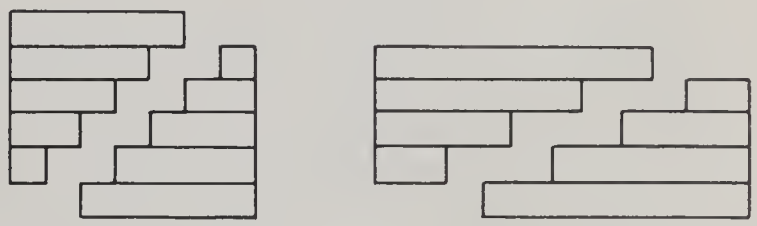
26. Rectangular solids may look different, yet have the same volume.



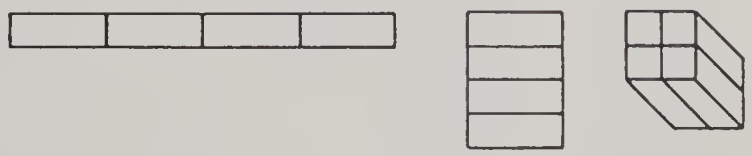
27. Staircases of various size steps can be constructed.



28. Sometimes two staircases of the same type can be made to "fit" when properly placed adjacent to each other.

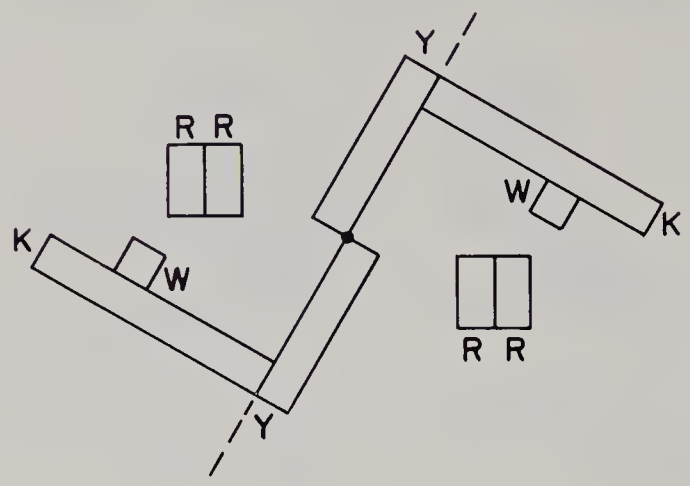


29. Trains of rods of the same color can be rearranged to form other floors, which in turn can be rearranged to form other rectangular solids.

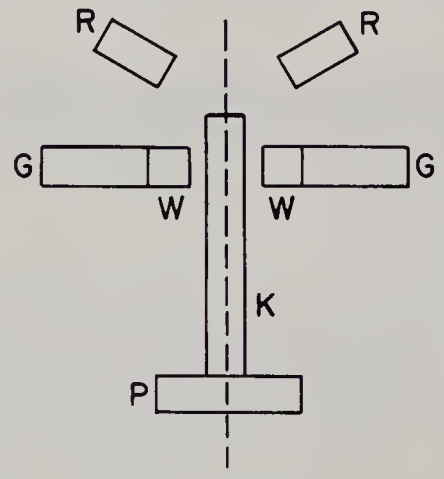


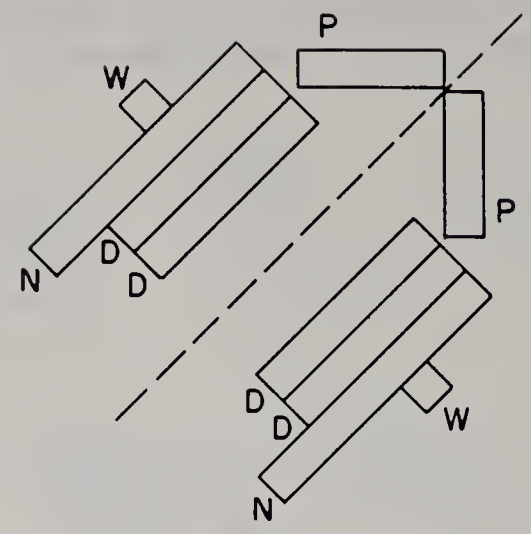
30 Rectangular solids consisting of rods of the same color can be rearranged to form floors, which in turn can be rearranged to form trains.

31. Patterns symmetrical with respect to a point can be constructed.

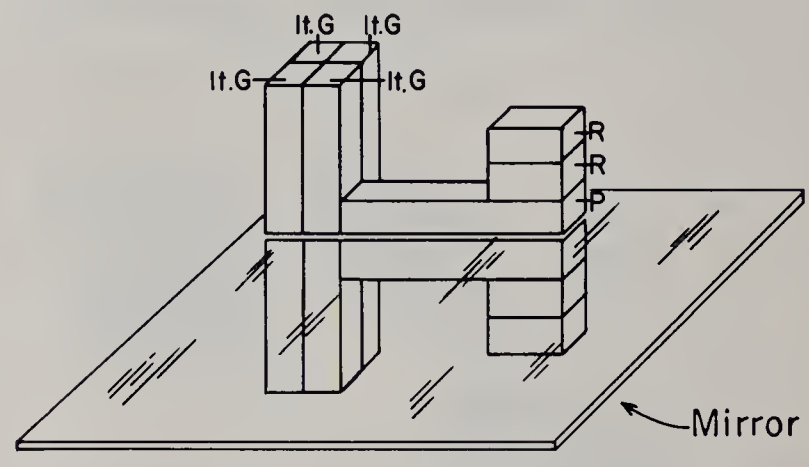


32. Patterns symmetrical with respect to a line can be constructed.

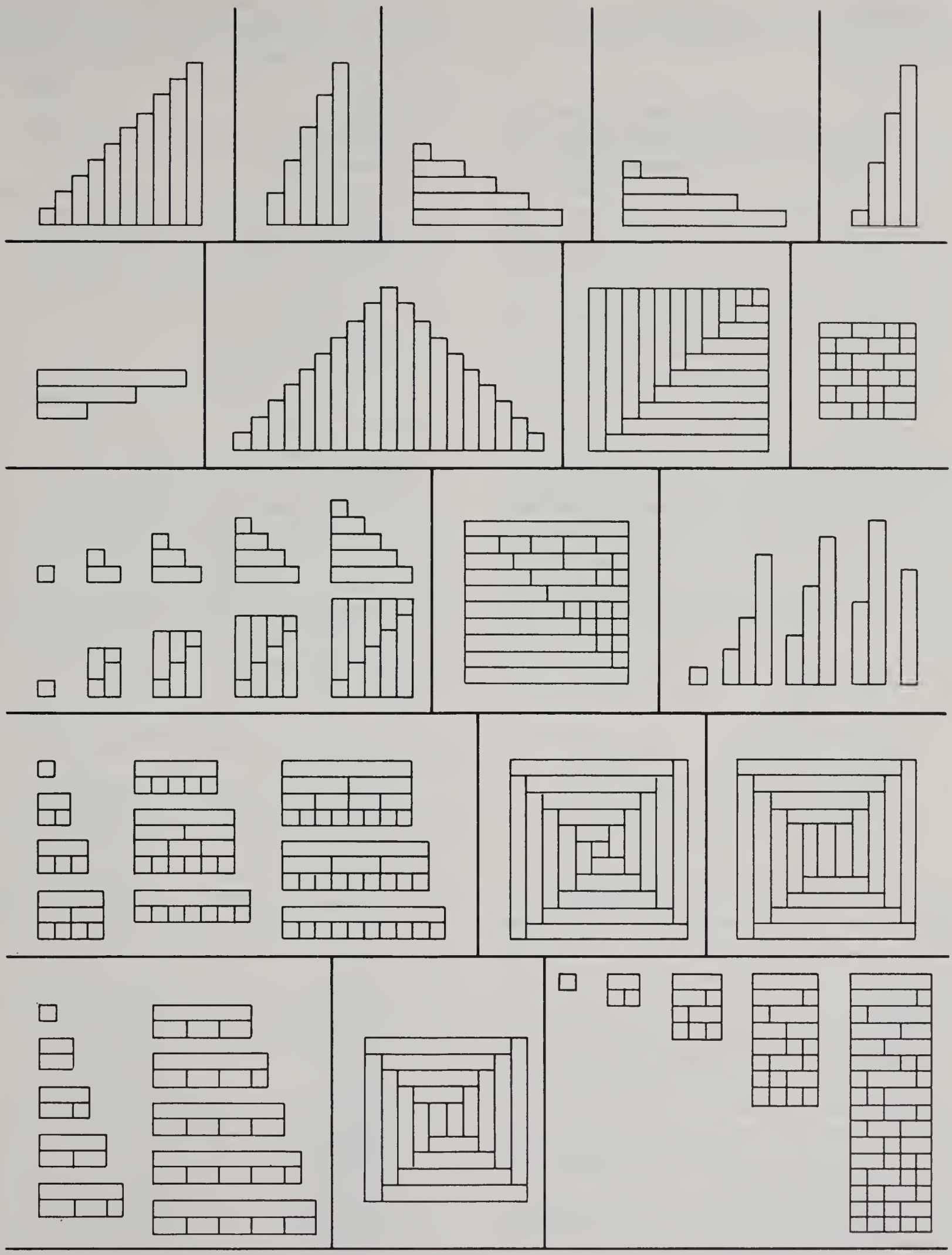




33. Patterns symmetrical with respect to a plane can be constructed.



34. For each rod of the same color there exists a unique (except for order) and finite decomposition pattern: each rod can be decomposed into a finite number of equivalent trains of rods. (The lower right-hand corner of the diagram on the next page shows the decomposition pattern for a white rod, a red rod, a light green rod, a purple rod, and a yellow rod.)

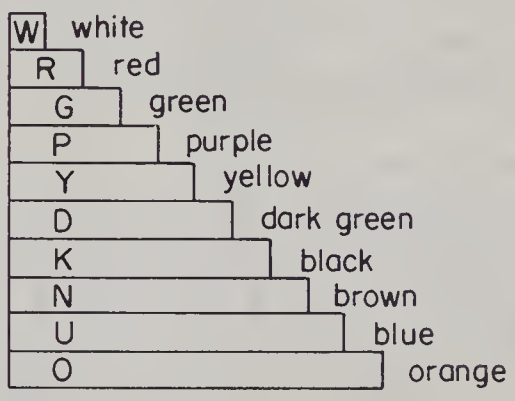


CUISENAIRE® RODS—SECTION II: ADDITION AND SUBTRACTION FACTS—FIRST DECADE

Up to this point you have been guiding your pupils toward gathering and organizing information acquired through actions on representational materials. You probably have encouraged them to use a variety of materials such as counters, disks, and beans. Your pupils most likely have also manipulated geometric regions such as triangular regions, square regions, rectangular regions, etc. They have worked with many model sets. Their work in the MATH WORKSHOP has afforded them ample opportunities and experiences to answer questions concerning "How many."

Comparison of Lengths

To help your pupils gain a more complete understanding of the concept of number, you will need to help them gather and organize information to make it possible for them to answer questions concerned primarily with "How much." Consider having your pupils use Cuisenaire® rods for this purpose.



You might wish to involve them in activities and discussions similar to the following:

Have each pupil place an orange rod flat on his desk.

Now a blue one.

A dialogue between you and your pupils might be:

Compare these two rods.

Which is longer—an orange rod or a blue rod?

How much longer is an orange rod than a blue rod? (One white rod longer.)

How much shorter is a blue rod than an orange rod? (One white rod shorter.)

Exchange the blue rod for a brown rod.

Which is longer—an orange rod or a brown rod?

How much longer is an orange rod than a brown rod? (One red rod longer, or two white rods longer.)

How much shorter is a brown rod than an orange rod? (One red rod shorter, or two white rods shorter.)

Encourage pupils to compare each rod in turn with an orange rod.

Have them discuss their findings with each other and help them summarize their information.

Now have them work with three rods at a time. For example, you might say to your pupils:

Place a blue rod, a black rod, and a purple rod flat on your desk.

Which rod is the longest?

Which rod is the shortest?

How much longer is a blue rod than a black one? (One red rod longer.)

How much shorter is a purple rod than a black one? (One light green rod shorter.)

Have your pupils form into groups of two or three or four. Suggest that they investigate a variety of triples of rods. Encourage group discussion. Allow ample time for this activity. Invite each group to share its findings with the other groups.

You might follow up the above activities with suggestions like these:

Place a yellow rod flat on your desk.

Make a two-car train that is shorter than the yellow rod. Can you make more than one such two-car train? (Yes.)

Make a two-car train that is longer than the yellow rod. Can you make more than one such two-car train? (Yes.)

Make a two-car train as long as a yellow rod. Can you make more than one such two-car train? (Yes.) Make as many two-car trains as you can that look different but that are long as a yellow rod.

You can make a record on the chalkboard of the information gathered by the class:

pupil construction

your record

first car	second car
P	W
W	P
G	R
R	G

Now have the pupils do the same thing for each of the other rods while you keep on the chalkboard

a written record of their findings. A summary of their investigations would eventually look like the following:

pupil construction

your record

white

1st	2nd
W	

red

1st	2nd
W	W

green

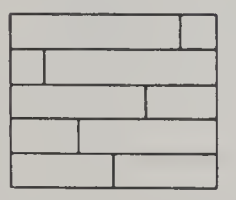
1st	2nd
R	W
W	R

purple

1st	2nd
G	W
W	G
R	R

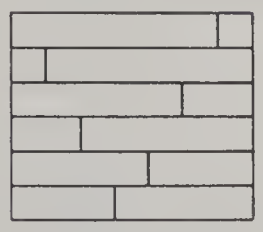
yellow

1st	2nd
P	W
W	P
G	R
R	G



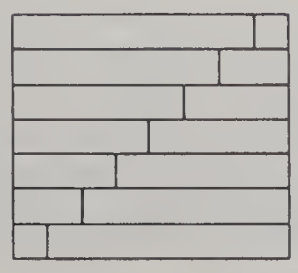
dark green

1st	2nd
Y	W
W	Y
P	R
R	P
G	G



black

1st	2nd
D	W
W	D
Y	R
R	Y
P	G
G	P



brown

1st	2nd
K	W
D	R
Y	G
P	P
G	Y
R	D
W	K

blue

1st	2nd
N	W
K	R
D	G
Y	P
P	Y
G	D
R	K
W	N

orange

1st	2nd
U	W
N	R
K	G
D	P
Y	Y
P	D
G	K
R	N
W	U

Ask pupils to investigate each pattern to find as many "hidden patterns" as they can. Have them discuss each pattern offered. Some typical discoveries are:

1. Each two-rod train or row in a pattern (except trains made of two rods of the same color) can be paired with another train in the pattern in which the rods in the train have been interchanged. For example, in the pattern of the yellow rod:

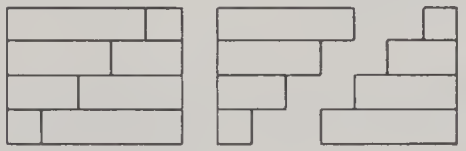
pupil construction

your record

Y

1st	2nd
P	W
W	P
G	R
R	G

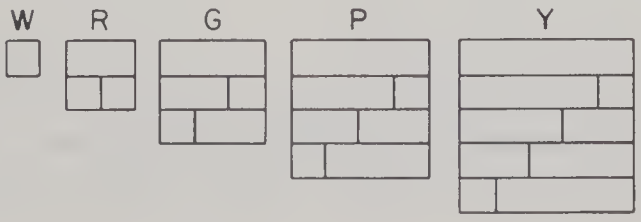
2. Each pattern can be separated into two staircases—while one staircase has smaller and smaller steps, the other has larger and larger steps:



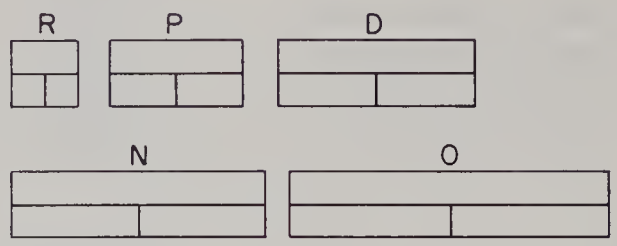
Y

1st	2nd
P	W
G	R
R	G
W	P

3. Each pattern is a square region if the one-rod train is included.



4. Starting with the pattern for the red rod, every other pattern has a row of two rods of the same color.



And so on.

You might follow this activity with the following instructions and questions for your pupils:

Place an orange rod flat on your desk.

Can you make a train as long as an orange rod, using only yellow rods? (Yes.)

Can you make a train as long as an orange rod, using only light green rods? (No.)

Make as many different looking trains as you can which are:

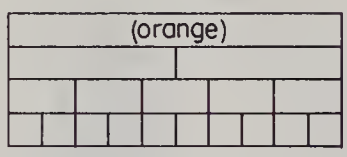
a) as long as an orange rod, and

b) made up of cars of just one color.

Repeat this for each of the rods.

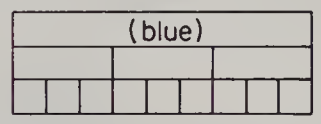
You can keep a record on the chalkboard:

pupil construction

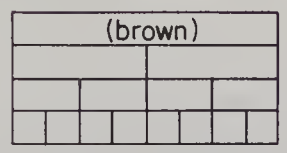


your record

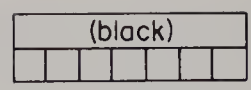
yes	no
O	
	U
	N
	K
	D
Y	
	P
	G
R	
W	



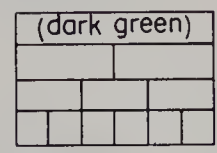
yes		U						G		W
no	O		N	K	D	Y	P		R	



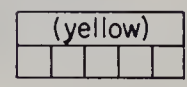
yes			N				P		R	W
no	O	U		K	D	Y		G		



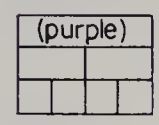
yes				K						W
no	O	U	N		D	Y	P	G	R	



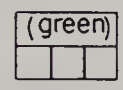
yes					D			G	R	W
no	O	U	N	K		Y	P			



yes						Y				W
no	O	U	N	K	D		P	G	R	

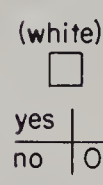


yes							P		R	W
no	O	U	N	K	D	Y		G		



yes								G		W
no	O	U	N	K	D	Y	P		R	

yes									R	W
no	O	U	N	K	D	Y	P	G		



yes										W
no	O	U	N	K	D	Y	P	G	R	

Establishing a Unit of Measure

Before proceeding to the next activity you may find it convenient to discuss with your pupils the notion of "Many names for the same thing." For

example: although in each case below you are the same person, you have many names.

To your pupils you are their teacher.

To your colleagues you are Miss, Mrs., or Mr.

To your parents you are their son or daughter.

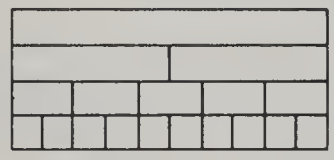
To your close friends you may be known by some special name.

In short, in any structured system of relationships each member of the system may be labeled differently depending upon the particular relationships considered.

Your pupils are no strangers to complex systems of relationships. They have already taken note of some of these before they entered your class. They know that to their grandparents they are grandchildren, to their parents they are children, to their brothers or sisters they are brothers or sisters, and to their uncles or aunts they are nieces or nephews. Yet in all the situations they are the same person—what changes is their relationship to others.

And so it is with the Cuisenaire® rods. As you will see, the rods do not change—only their relationships to other rods change.

Have the class reconstruct the pattern of the orange rod in which rods of *one color only* were used to build trains as long as the orange rod.



Ask questions similar to the following:

What can we say about all the trains in the pattern? (All trains are the same in length:

a train of two yellow rods is as long as a train of one orange rod,

a train of five red rods is as long as a train of one orange rod,

a train of ten white rods is as long as a train of one orange rod,

a train of two yellow rods is as long as a train of five red rods,

a train of two yellow rods is as long as a train of ten white rods.)

We know that a train of two yellow rods is as long as an orange rod. If we change a yellow rod's name from *yellow* to *one*, what will be the new name of an orange rod? (*Two*—because a train of two ones is as long as an orange rod.) Record the information on the board as follows:

old name	yellow	orange
new name	1	2

We know that a train of five red rods is as long as an orange rod. If we change a red rod's name from *red* to *one*, what will be the new name of an orange rod? (*Five*, since a train of five ones is as long as an orange rod.)

old name	red	orange
new name	1	5

Representing the Basic Facts of Arithmetic

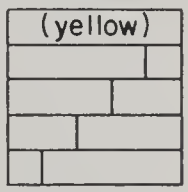
The activites just described are both necessary and desirable. You are now in a position to help your pupils become further acquainted with at least the basic addition facts of the first decade. Activities similar to the following are now appropriate. Sample directions and questions are:

Place a yellow rod flat on your desk.

Build as many two-car trains as you can which

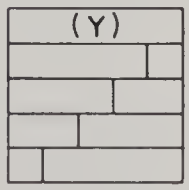
a) look different, and
b) are as long as the yellow rod.

When the pupils have constructed their patterns, sketch the patterns on the chalkboard. Record the patterns, using letter symbols as follows:



1st	2nd
Y	
P	W
G	R
R	G
W	P

Ask the pupils what the record would look like if we called a white rod *one*. After the class agrees on the new name of each rod, invite some pupils to translate the old record into a new one, using the new names. Their records should look like this:



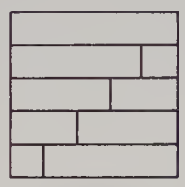
1st	2nd
5	
4	1
3	2
2	3
1	4

Number Sentences

Discuss ways of writing (or saying) sentences about each train, using the new names.

If the pupils agree to consider the construction

of a train as representing the operation of addition, they will probably offer the following sentences:



number sentences

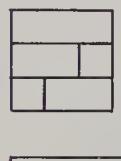
5 + 0 = 5
4 + 1 = 5
3 + 2 = 5
2 + 3 = 5
1 + 4 = 5

Ask pupils to read each of the sentences written. (You might want to label these sentences as "number sentences.")

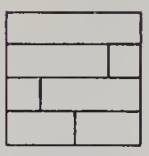
Work through each pattern with the children until the following conclusions are reached:

1 + 0 = 1

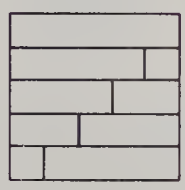
2 + 0 = 2
1 + 1 = 2



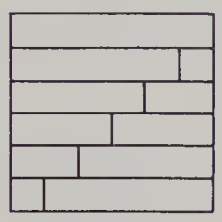
3 + 0 = 3
2 + 1 = 3
1 + 2 = 3



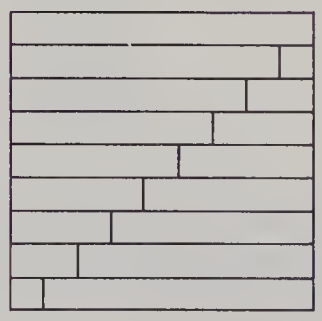
4 + 0 = 4
3 + 1 = 4
1 + 3 = 4
2 + 2 = 4



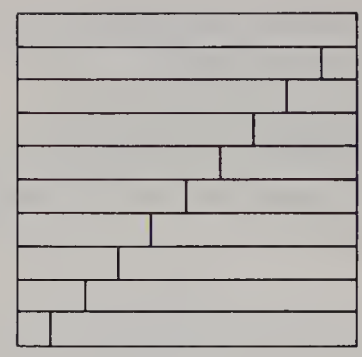
5 + 0 = 5
4 + 1 = 5
3 + 2 = 5
2 + 3 = 5
1 + 4 = 5



6 + 0 = 6
5 + 1 = 6
4 + 2 = 6
3 + 3 = 6
2 + 4 = 6
1 + 5 = 6

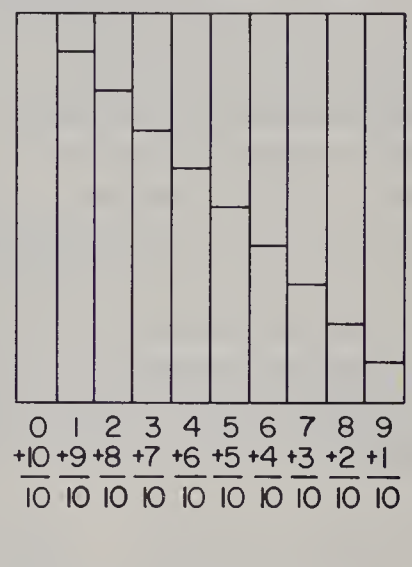


9 + 0 = 9
8 + 1 = 9
7 + 2 = 9
6 + 3 = 9
5 + 4 = 9
4 + 5 = 9
3 + 6 = 9
2 + 7 = 9
1 + 8 = 9



10 + 0 = 10
9 + 1 = 10
8 + 2 = 10
7 + 3 = 10
6 + 4 = 10
5 + 5 = 10
4 + 6 = 10
3 + 7 = 10
2 + 8 = 10
1 + 9 = 10

By rotating each pattern a quarter turn clockwise or counterclockwise, the column form for the addition facts is suggested. Here is what the rotated pattern for 10 looks like:



0	1	2	3	4	5	6	7	8	9
+10	+9	+8	+7	+6	+5	+4	+3	+2	+1
10	10	10	10	10	10	10	10	10	10

Similar rotations on each of the other patterns will yield the column form of the addition facts for 9, 8, 7, 6, 5, 4, 3, 2, and 1.

This sequence of activitïes in which (1) pupils construct the pattern, (2) pupils discuss the pattern, and (3) you write about the pattern (on the chalkboard) in number sentence form or column form, could be reversed as follows:

(1) You write about a pattern (on the chalkboard) in number sentence form or column form, (2) pupils discuss the writtern pattern, and (3) pupils construct the pattern with rods by responding to the written notation on the chalkboard.

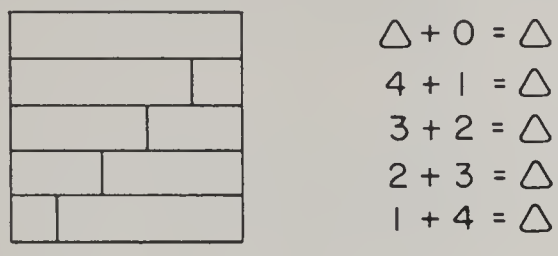
Prepare Dittoed sheets which contain the following tasks as further enrichment:

1. A diagram of a pattern drawn to size so that pupils can place rods on the pattern to complete it.
2. A record, using color symbols (letters).
3. A record, using numbers.
4. Number sentences or column forms depending upon whether a pattern is arranged horizontally or vertically.

The Missing Sum Game

Groups of three or four pupils select a group leader. The group leader constructs a complete pattern of two-car trains for any of the ten rods. Upon completion, the other group members must determine which rod would be the single-car train for the pattern built by the leader.

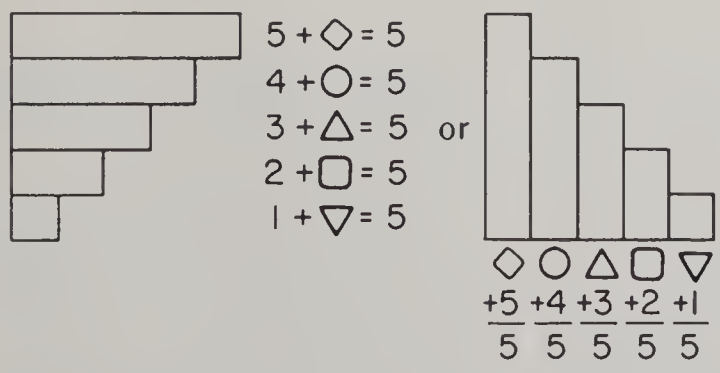
The group leader (or you) then writes on the chalkboard as many open sentences as are appropriate for that pattern, using a frame as the place holder for the missing sum.



The group then decides what the missing sum must be. Each group might work on a different pattern. The appropriate open sentences are written on the chalkboard. Now each group must make each open sentence true (and be prepared to prove it by the rods if anyone from the original group that worked on the pattern requires it).

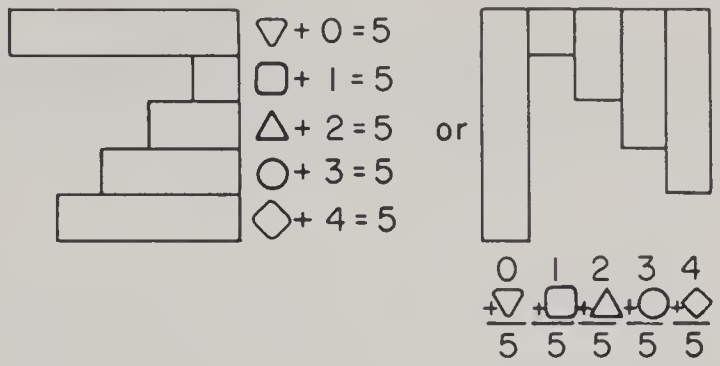
The Missing Addend Game

The group leader constructs a complete two-car train pattern for any rod the group decides to work on. The leader then removes the second rod from each train. Each member of the group then decides which is the missing addend in each train. The group recorder (or you) then writes the appropriate open sentences on the chalkboard, using a frame as a place holder for the missing addend:



The group then decides what the missing addend must be. Now each group must make each open sentence true (and be prepared to prove it by the rods if anyone from the original group that worked on the pattern requires it).

The game can be modified by constructing a complete pattern of two-car trains and then removing the first addend instead of the second:



The Missing Sum or Missing Addend Game

a) Each member of the group selects a row from any complete two-car train pattern. The other group members must formulate the appropriate open number sentence. A group recorder (or you) writes the open sentence on the chalkboard. The group then makes the open sentence true.

b) Each member of a group selects at random from a complete two-car pattern the single-car train and one rod from any other train in the pattern. The group formulates the appropriate open number sentence. The recorder (or you) writes the open number sentence on the chalkboard. The group then completes the open number sentence by making it true.

c) Modify the previous games (a, b) by having group members complete the open sentences by making them false.

d) You write random number sentences on the board and ask various pupils to read them. You then ask the pupils to prove the sentences either true or false (using the rods when necessary).

	true	false
2 + 3 = 5	✓	
10 + 4 = 5		✓
7 + 1 = 8	✓	
3 = 2 + 2		✓
6 + 5 = 10		✓
1 + 2 = △		neither true nor false

e) You write on the chalkboard the random open number sentences (for missing sums). Pupils will complete the open sentences to make them

true sentences (or false sentences if you so direct). Pupils prove their statements by the rods when necessary.

$2 + 5 = \square$
$3 + 6 = \triangle$
$1 + 0 = \triangledown$

or

$$\begin{array}{ccc} 2 & 3 & 1 \\ +5 & +6 & +0 \\ \hline \square & \triangle & \triangledown \end{array}$$

f) Conduct similar activities for random selections of open number sentences with missing addends.

$5 + \square = 10$
$6 + \square = 7$
$4 + \triangle = 4$
$\triangledown + 7 = 8$
$\diamond + 2 = 4$
$\square + 3 = 9$

or

$$\begin{array}{cccc} \square & 2 & 4 & \square \\ +3 & +\triangle & +\triangledown & +7 \\ \hline 5 & 9 & 10 & 7 \end{array}$$

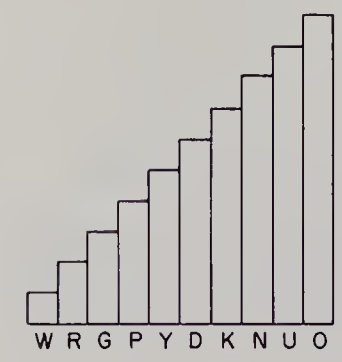
You can then follow up these activities with Dittoed materials for individual study and help.

CUISENAIRE® RODS—SECTION III: ADDITION AND SUBTRACTION FACTS—SECOND DECADE

(Consider the white rod's name to be "one" for all activities discussed in this section.)

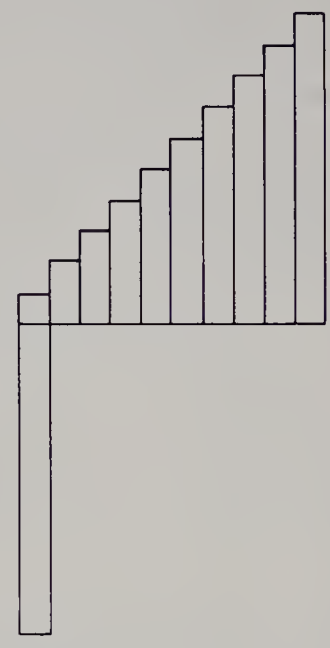
Naming Numbers of the Second Decade

Have your pupils select one rod of each color to build a staircase from W to O as shown below in the left-hand diagram.

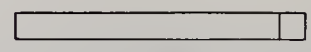


Have different pupils lead the class in counting up and down the staircase.

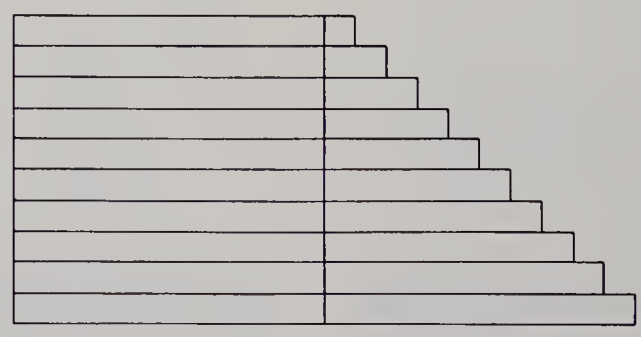
Now have each pupil place an orange rod below the white rod in the staircase:



Ask what number name should be assigned to the train O + W. Start the following record on the board and discuss it with the class:

Rod Pattern	Number name	Word description	Standard word name
	10 + 1	ten plus one	eleven

Have pupils place an orange rod under each of the other rods in the staircase while you record and discuss the appropriate information for each shift. Your record will eventually look like the diagram below on the right.

Rod Pattern	Number name	Word description	Standard word name
	10 + 1	ten plus one	eleven
	10 + 2	ten plus two	twelve
	10 + 3	ten plus three	thirteen
	10 + 4	ten plus four	fourteen
	10 + 5	ten plus five	fifteen
	10 + 6	ten plus six	sixteen
	10 + 7	ten plus seven	seventeen
	10 + 8	ten plus eight	eighteen
	10 + 9	ten plus nine	nineteen
	10 + 10	ten plus ten	twenty

Have the class form three-member practice teams. Each team engages in the following activity:

A constructs a train of an orange rod and some other rod.

B attempts to give the appropriate number name to A's train.

C checks A's construction and B's number name, referring to the chalkboard record when necessary.

The game is played until all team members have had successful experiences at being constructors, namers, and checkers.

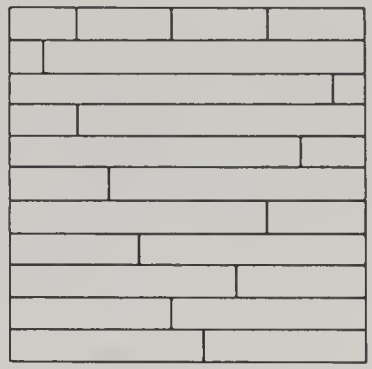
The Fewest Rods Game (1)

Rule 1. Two or more rods are to be selected so that a train can be formed which is longer than an orange rod but not longer than a train of two orange rods.

Rule 2. Other trains are to be constructed so that each is equal in length to the first train and is constructed with the fewest rods possible.

For example, suppose that someone builds the following train:

Below is a diagram of the possibilities for trains with the fewest rods—each of which is equal in length to the first train:



To play the game, pupils form three-member teams.

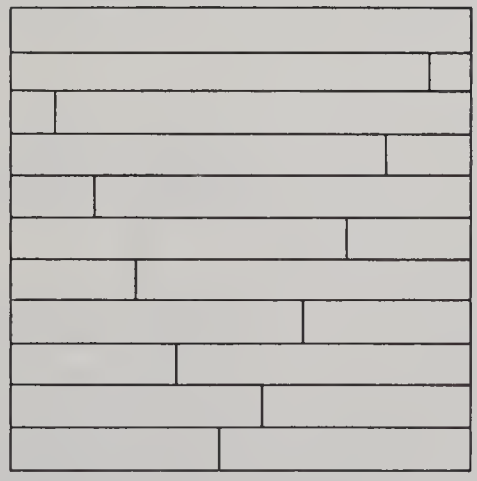
A constructs a train to meet the requirements of Rule 1.

B constructs as many trains as possible which meet the requirements of Rule 2.

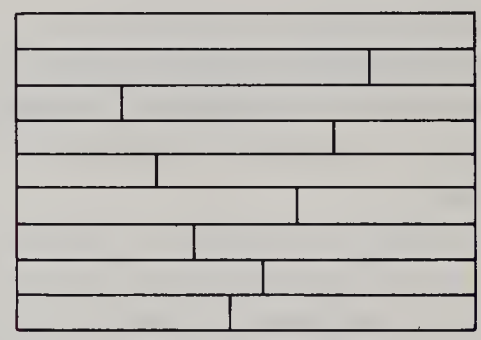
C checks to see that A meets the requirements of Rule 1 and that B meets the requirements of Rule 2.

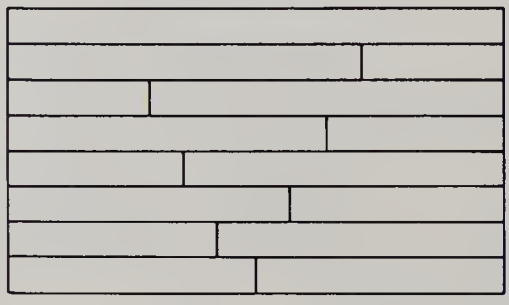
Games are played so that each team member has had a chance to be a constructor of trains for Rule 1 and Rule 2, as well as a checker.

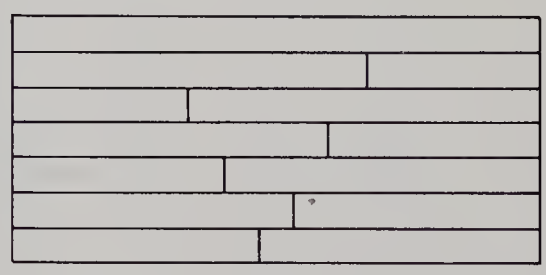
After the class has had ample time to play the game, invite your pupils to help you summarize their experiences on the chalkboard. Your summary will eventually look like the one shown below and on the next two pages.

Pattern for eleven	Number name	Word description	Standard word name
			eleven
	10 + 1	ten plus one	eleven
	1 + 10	one plus ten	eleven
	9 + 2	nine plus two	eleven
	2 + 9	two plus nine	eleven
	8 + 3	eight plus three	eleven
	3 + 8	three plus eight	eleven
	7 + 4	seven plus four	eleven
	4 + 7	four plus seven	eleven
	6 + 5	six plus five	eleven
	5 + 6	five plus six	eleven

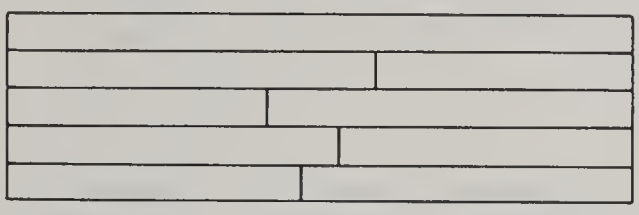
Pattern for twelve	Number name	Word description	Standard word name
			twelve
	10 + 2	ten plus two	twelve
	2 + 10	two plus ten	twelve
	9 + 3	nine plus three	twelve
	3 + 9	three plus nine	twelve
	8 + 4	eight plus four	twelve
	4 + 8	four plus eight	twelve
	7 + 5	seven plus five	twelve
	5 + 7	five plus seven	twelve
	6 + 6	six plus six	twelve

Pattern for thirteen	Number name	Word description	Standard word name
			thirteen
	10 + 3	ten plus three	thirteen
	3 + 10	three plus ten	thirteen
	9 + 4	nine plus four	thirteen
	4 + 9	four plus nine	thirteen
	8 + 5	eight plus five	thirteen
	5 + 8	five plus eight	thirteen
	7 + 6	seven plus six	thirteen
	6 + 7	six plus seven	thirteen

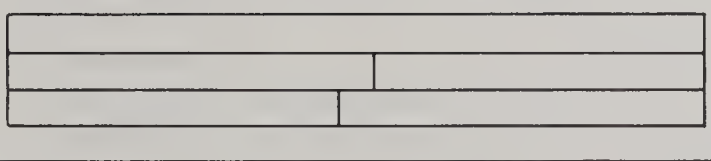
Pattern for fourteen	Number name	Word description	Standard word name
			fourteen
	10 + 4	ten plus four	fourteen
	4 + 10	four plus ten	fourteen
	9 + 5	nine plus five	fourteen
	5 + 9	five plus nine	fourteen
	8 + 6	eight plus six	fourteen
	6 + 8	six plus eight	fourteen
	7 + 7	seven plus seven	fourteen

Pattern for fifteen	Number name	Word description	Standard word name
			fifteen
	10 + 5	ten plus five	fifteen
	5 + 10	five plus ten	fifteen
	9 + 6	nine plus six	fifteen
	6 + 9	six plus nine	fifteen
	8 + 7	eight plus seven	fifteen
	7 + 8	seven plus eight	fifteen

Pattern for sixteen	Number name	Word description	Standard word name
			sixteen
	10 + 6	ten plus six	sixteen
	6 + 10	six plus ten	sixteen
	9 + 7	nine plus seven	sixteen
	7 + 9	seven plus nine	sixteen
	8 + 8	eight plus eight	sixteen

Pattern for seventeen	Number name	Word description	Standard word name
			seventeen
	10+7	ten plus seven	seventeen
	7+10	seven plus ten	seventeen
	9+8	nine plus eight	seventeen
	8+9	eight plus nine	seventeen

Pattern for eighteen	Number name	Word description	Standard word name
			eighteen
	10+8	ten plus eight	eighteen
	8+10	eight plus ten	eighteen
	9+9	nine plus nine	eighteen

Pattern for nineteen	Number name	Word description	Standard word name
			nineteen
	10+9	ten plus nine	nineteen
	9+10	nine plus ten	nineteen

Pattern for twenty	Number name	Word description	Standard word name
			twenty
	10+10	ten plus ten	twenty

The Fewest Rods Game (2)

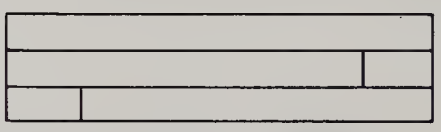
Rule 1. Select two or more rods that can form a train which is longer than an orange rod but not longer than a train of two orange rods.

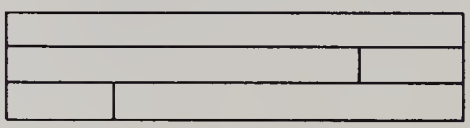
Rule 2. Construct other trains so that each train is equal in length to the first and at least one car of each new train is an orange rod.

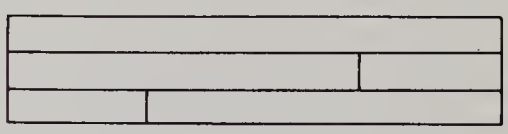
To play, the same pupils form their three-member play teams and assume roles similar to those described in the Fewest Rods Game (1).

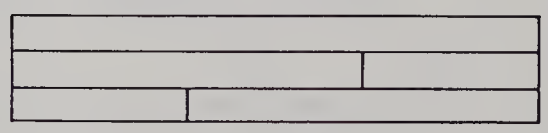
Have the class help you summarize their experiences on the board. Your summary will eventually look like this:

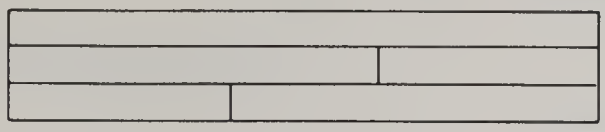
Pattern for eleven	Number name	Word description	Standard word name
	10 + 1 1 + 10	ten plus one one plus ten	eleven eleven eleven

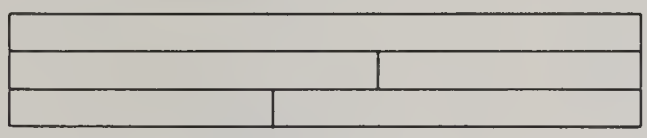
Pattern for twelve	Number name	Word description	Standard word name
	10 + 2 2 + 10	ten plus two two plus ten	twelve twelve twelve

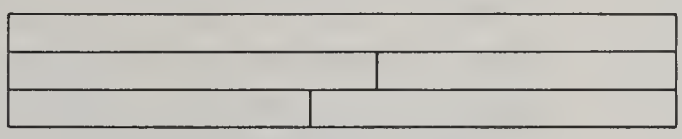
Pattern for thirteen	Number name	Word description	Standard word name
	10 + 3 3 + 10	ten plus three three plus ten	thirteen thirteen thirteen

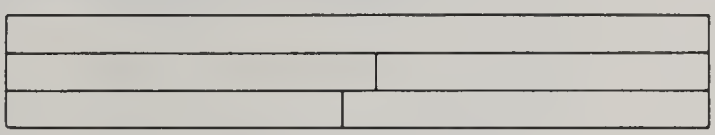
Pattern for fourteen	Number name	Word description	Standard word name
	10 + 4 4 + 10	ten plus four four plus ten	fourteen fourteen fourteen

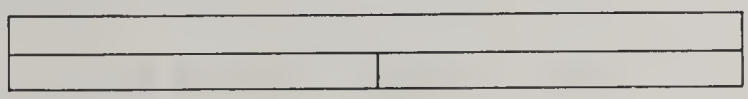
Pattern for fifteen	Number name	Word description	Standard word name
	10 + 5 5 + 10	ten plus five five plus ten	fifteen fifteen fifteen

Pattern for sixteen	Number name	Word description	Standard word name
	10 + 6 6 + 10	ten plus six six plus ten	sixteen sixteen sixteen

Pattern for seventeen	Number name	Word description	Standard word name
	10 + 7 7 + 10	ten plus seven seven plus ten	seventeen seventeen seventeen

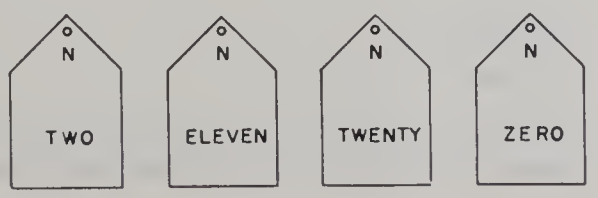
Pattern for eighteen	Number name	Word description	Standard word name
	10 + 8 8 + 10	ten plus eight eight plus ten	eighteen eighteen eighteen

Pattern for nineteen	Number name	Word description	Standard word name
	10 + 9 9 + 10	ten plus nine nine plus ten	nineteen nineteen nineteen

Pattern for twenty	Number name	Word description	Standard word name
	10 + 10	ten plus ten	twenty twenty

The Number Tags Game

Prepare sets of tags with the standard number word names (0 through 20) printed on them as in the sketch below:



Pupil teams of four members can be formed to play the following game:

A selects one of the number tags to show to pupils B, C, and D.

B selects a set of rods he estimates he will need to build a train named by the number tag and places them on the table end to end.

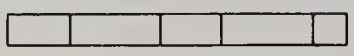
C checks B's construction.

D records the results.

Pupils play the game so that each has several chances at playing the role of a number tag selector, train constructor, checker, and recorder. A partial record of a few rounds of the game would look like this:

A (selects)

B (constructs)



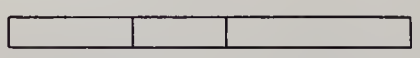
C (checks)

D (records)

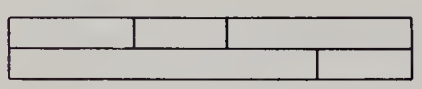
2 + 3 + 2 + 3 + 1 = 10 + 1

A (selects)

B (constructs)



C (checks)



D (records)

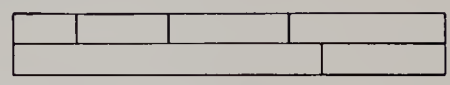
4 + 3 + 6 = 10 + 3

After pupils have had sufficient experiences with this game, you might try this variation by reversing the sequence of actions as follows. You write on the board one row of a record. The class constructs the two trains described by the record, and finally someone is asked to select the tag that goes with the construction:

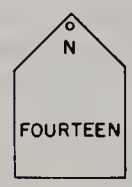
Record

2 + 3 + 4 + 5 = 10 + 4

Class constructions



Number tag selection



The Maximum Rods Game

Rule 1. One or more rods are selected to construct a train as long as or longer than a white rod, but not longer than a train of two orange rods.

Rule 2. Another train is to be constructed so that its length is equal to that of a train constructed in accordance with Rule 1 and is made up of the maximum number of rods possible.

Four-member teams can be formed to play this game.

Pupil A constructs a train in accordance with Rule 1.

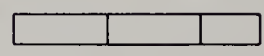
Pupil B constructs a train in accordance with Rule 2.

Pupil C checks A's work and B' work.

Pupil D records the results:

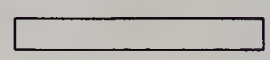
1st. Round

A (constructs)



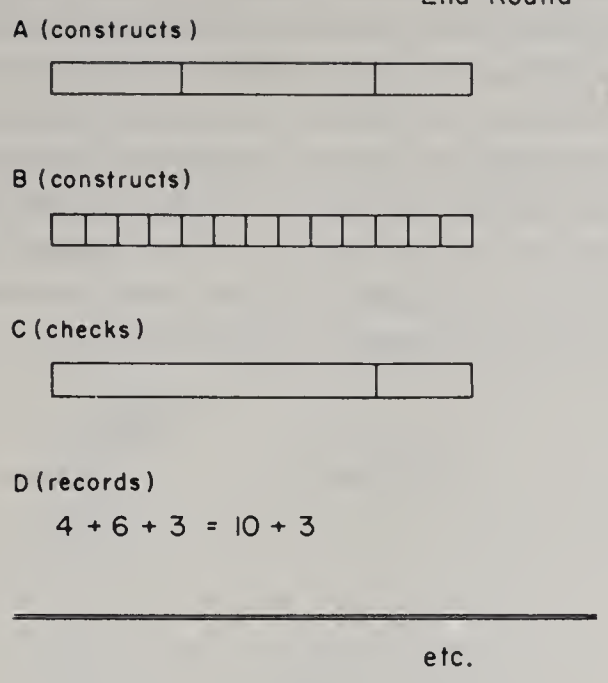
B (constructs)

C (checks)



D (records)

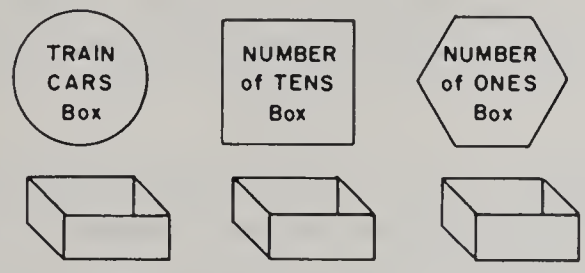
3 + 3 + 2 = 8



Assuming all pupils have had considerable experience with this game, they are now ready for the Three Boxes game.

The Three Boxes Game

Arrange your demonstration table in front of the chalkboard. Place on it three boxes with lids containing the following information on them:



Rule 1. Any set of rods placed in the "Train Cars Box" must be such that the length of the corresponding train is not longer than a train of two orange rods.

Rule 2. Only orange rods may be placed in the "Number of Tens Box."

Rule 3. Only white rods may be placed in the "Number of Ones Box."

The game can be played with the entire class as follows:

Invite a pupil to select from the pile on your desk a set of rods he estimates will satisfy Rule 1 and place the rods in the "Train Cars Box." Another pupil is selected to call out to the class the number names of each rod in the "Train Cars Box." After each pupil in the class has constructed at his desk the train described, the pupils construct another train using only white rods. Finally, they construct a third train in which they are to use as many orange rods as possible so long as the train is as long as the train of white rods. The four types of situations the pupils will encounter are:

a) The train of white rods is shorter than an orange rod. If this is so, then the number of tens used would be zero and the number of white rods used would be some number greater than zero but less than ten.

b) The train of white rods is the same length as an orange rod. If this is so, then the number of tens would be one and the number of white rods left over would be zero.

c) The train of white rods is longer than one orange rod but shorter than two orange rods. If this is so, then the number of tens is one and the number of white rods left over would be some number greater than zero but less than ten.

d) The train of white rods is as long as a train of two orange rods. If this is so, then the number of tens is two and the number of white rods left over is zero.

Start a record of the game on the chalkboard. A record after six rounds of play might look like the following:

TRAIN CARS Box	NUMBER of TENS Box	NUMBER of ONES Box
2 + 3 + 4 + 5 = 10 + 4	1	4
4 + 2 + 1 = 7		7
6 + 4 + 5 + 2 = 10 + 7	1	7
7 + 4 = 10 + 1	1	1
1 + 3 + 2 = 6		6
4 + 5 + 6 + 5 = 10 + 10	2	0

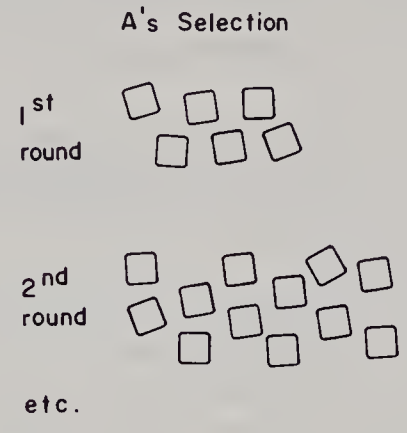
etc.

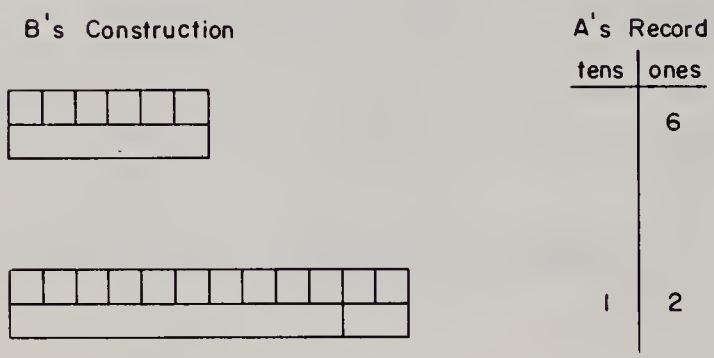
Since one of the objectives of this game is to introduce pupils to our decimal system of numeration, several variations of the game will be of interest to your pupils. For example, two-member pupil teams play as follows:

A places a set of white rods in a container. B empties the container and makes a train of the white rods. B then constructs a train equal in length to the train of white rods, using as many orange rods as possible and any other rod necessary

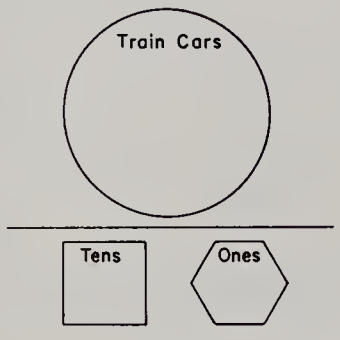
to complete the length of the train of white rods. A then records the number of tens and ones on his record sheet.

Pupils change roles for as many rounds of this game as seems desirable. A record of several rounds of this game would look like the following:

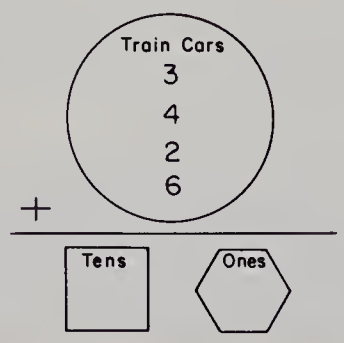




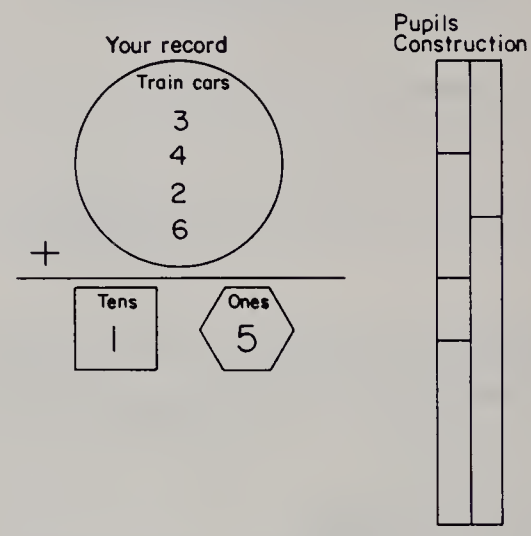
Sketch the following on the board:



In the circle list several numbers with totals less than, or equal to, 20.



Ask the class to construct a train, using rods named by the numbers in the circle. Pupils then construct a train of as many orange rods and whatever rods they need to complete the train. Someone is selected to tell the class the number of orange rods he used and the number of white rods needed to complete the train. You record the results on the chalkboard. A diagram of the pupil construction and your record might look like this:



After the class has had sufficient experience with the chalkboard records, prepare oak tag sheets with the diagram of the circular, square, and hexagonal lids arranged in column addition form. Two-member teams can be formed, so that each team has an oak tag sheet and rods to play the game.

A places a train of rods in the circle in such a way that its length is less than, or equal to, that of two orange rods.

B constructs a train as long as A's train, using orange rods and/or another rod.

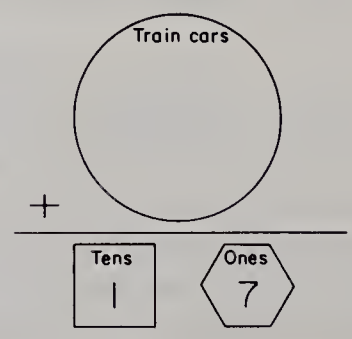
A records the addends on the chalkboard.

B records the number of tens and ones.

Pupils can shift roles and play the game as often as seems desirable.

A Reconstruction Game

Sketch the following on the board:



Ask the class to construct the train suggested by the given information. Now ask the pupils to construct as many trains as they can which look different but are as long as the first train they constructed. Invite various pupils to write their list of addends (or number names of cars in a train) on the board. Each list that is written is to be checked by the rest of the class.

Pupil teams of three or more members can be formed to play this reconstruction game. Supply each team with a Dittoed sheet containing one (or more) reconstruction problems. All team members are to participate in constructing as many trains as possible in the time allotted. The trains should look different but be of the same length. One member of the team keeps a record of the team's efforts. Each team then makes a team report to the rest of the class. Each construction reported by a team can be checked by the rest of the class. A team report might look like the following:

		5			
1	3	2		3	
2	3	2		4	1
3	3	2	5	5	2
4	3	2	5	2	6
5	3	2	5	2	7
2/17	2/17	2/17	2/17	1/17	1/17

The Game of Clues

Since clue games range in difficulty from simple to complex, we will present here a few of the many possibilities available to you.

SAMPLE I.

Clue 1. The number name of the train is *one.*

Question. Do you know the train I'm thinking of?

Answer. The answer is obviously one white rod.

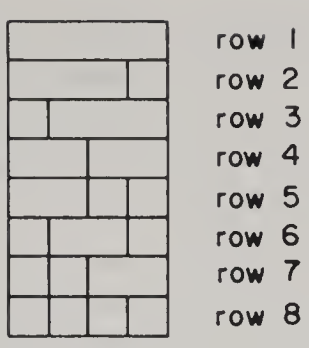
SAMPLE II.

Teacher: Do you know the number name of the train I'm thinking of?

Pupils: Not yet.

Teacher (Clue 1): The number name of the train is *four.* Do you know the train I'm thinking of?

Pupils: The thoughtful pupil will still reply with "not yet," since there are eight possibilities for such a train:



Teacher (Clue 2): The train has just three cars in it. Do you know which train I'm thinking of?

Pupils: The thoughtful pupil will rule out rows 1, 2, 3, 4, and 8 and report that the train consists of a two and two ones. The very thoughtful pupil will say, "I need another clue."

Teacher (Clue 3): The first car in the train is one.

Pupils: A not-so-careful pupil might say the train is 1 + 2 + 1. A very careful pupil will say, "I need one more clue."

Teacher (Clue 4): The last car in the train is two.

Pupils: All pupils will say, "The train is 1 + 1 + 2."

SAMPLE III.

Teacher (Clue 1): The number name of the train is *eighteen.*

Pupils: We need another clue.

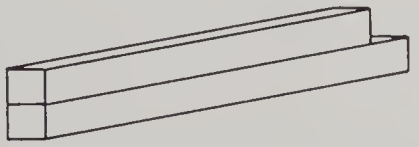
Teacher (Clue 2): The train is made up of two cars with the same name.

Pupils: Most pupils who use the rods will quickly tell you that they know the train you're thinking of by saying, "We don't need any more clues—the train is 9 + 9."

Clue games of the sort just described will provide your pupils with valuable beginning experiences in deductive reasoning.

The What's-the-Difference? Game

Invite the class to place an orange rod and a blue rod on their desks. Have them cover the orange rod with the blue rod as shown below:



A rod which is as long as the uncovered portion of the orange rod is the difference between the

orange rod and the blue rod. Since the covering action is quite different from that of making a train, the notation should reflect the new action. Using number names and the subtraction sign, you can record their findings as $10 - 9 = 1$ and say, "Ten minus nine equals one."

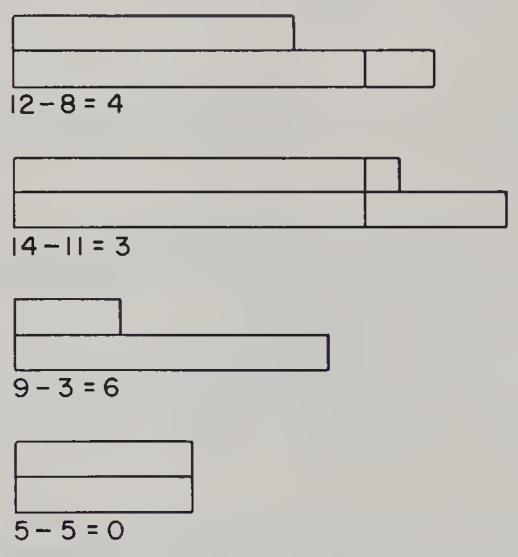
Ask the class to construct trains for seventeen and thirteen. Have them cover the seventeen train with the thirteen train and ask, "What's the difference?" Pupils will find that the uncovered portion of the seventeen train is as long as a four train. Write on the board $17 - 13 = 4$ and say, "Seventeen minus thirteen equals four."

After pupils have had sufficient experiences of the type just described, they can form two-member teams and play the What's-the-Difference? game.

Pupil A constructs a train.

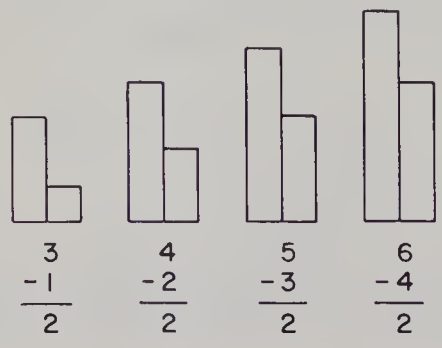
Pupil B constructs a train.

The pupil who constructs the smaller train covers the larger train with his train. Both must agree on the difference. Both record their actions. A partial record of a possible team activity shows the following:



The Family of Differences Game

One of the interesting observations pupils make as a result of playing What's-the-Difference? games is that frequently a variety of constructions have the same difference:



Once this observation is made and adequately explored for verification, pupil teams can be formed to play the Family of Differences game as follows:

A constructs one member of a family.

B verifies and records A's actions.

B constructs another member of the same family.

A verifies and records B's actions.

A constructs another member of the same family.

B verifies and records A's actions.

B constructs another member of the same family.

A verifies and records B's actions.

And so on.

A time limit can be placed on the game so that at a given signal all teams must begin constructing a new family of differences. A summary of a game would reveal the following information:

Family	0	1	2	3	.....	20
	0-0	1-0	2-0	3-0		20-0
	1-1	2-1	3-1	4-1		
	2-2	3-2	4-2	5-2		
	3-3	4-3	5-3	6-3		
	4-4	5-4	6-4	7-4		
	⋮	⋮	⋮	⋮		
	20-20	20-19	20-18	20-17		

Pupils can extend any or all families as each new decade is explored.

Experiences with Family of Differences games will provide your pupils with necessary information for developing several different approaches to the subtraction algorithm later on.

The Sum and Difference Game

A variation of the Game of Clues is the Sum and Difference game. In this game, pupils are to discover two numbers, given only two clues—their sum and their difference. Prepare the class for this game as follows:

Have them place an orange rod and a blue rod on their desks. Ask: "What is the sum of 10 and 9?" Pupils will form the train of orange and blue and identify it as nineteen. Write on the board: $10 + 9 = 19$ Ask: "What's the difference between 10 and 9?" Pupils will cover the

orange rod with the blue rod and identify the difference as 1. Write on the board:

$$10 - 9 = 1$$

Now tell the class you are thinking of two numbers and that they can find these numbers with just two clues from you. Write on the board:

Sum: 19
Difference: 1

"What are the numbers I'm thinking of?" It won't be long before someone tells you that your numbers must be 10 and 9.

Have the class take another pair of rods; for example, brown and red. Have them form a train and ask, "What is the sum of 8 and 2?" Have them cover the brown rod with the red rod and ask, "What is the difference between 8 and 2?" Write on the board:

$$8 + 2 = 10$$
$$8 - 2 = 6$$

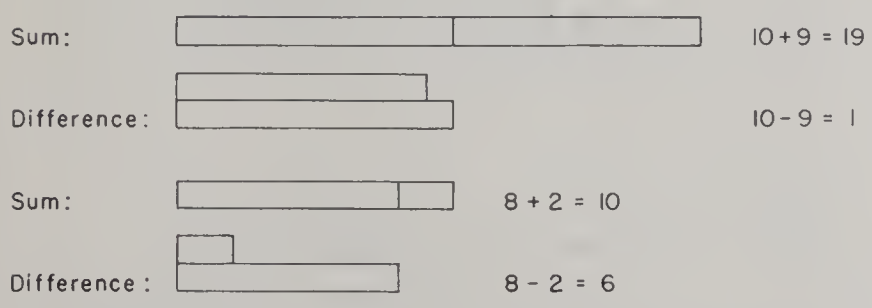
Ask if these two sentences describe the game they just played.

Now tell the class that you're thinking of two numbers:

Sum: 10
Difference: 6

"What are the two numbers I'm thinking of?" Pupils will readily tell you that your numbers must be 8 and 2.

A sketch of the two Sum and Difference games just played would show the following:



After several practice rounds with you, the class is ready for the Sum and Difference game. Two-member teams can be formed to gain readiness experiences for the game as follows:

A constructs a train.

B constructs a train.

A forms the sum of the two trains; B records A's actions.

B forms the difference between the two trains; A records B's actions.

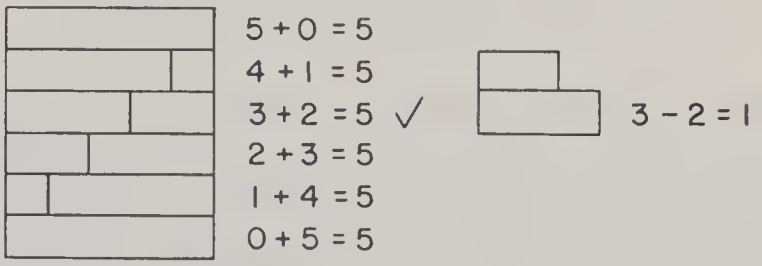
Pupils need ample experience with these games before you reverse the situation as follows:

Write on the board:

Clue 1. Sum: 5
Clue 2. Difference: 1

"What are the two numbers?"

By trial and error pupils will come to realize that although there are six possibilities for the sum 5, using two whole numbers, only one of these possibilities will simultaneously yield a difference of 1, namely, 3 and 2.



Guide the class through a number of these games before presenting them with the following:

Sum: □ + △ = 7
Difference: □ - △ = 1

" □ and △ are holding places for which numbers? Write numbers in the place holders which will make both sentences true simultaneously."

After guiding the class through several rounds of this activity, write games on the board for the pupils to work out for themselves:

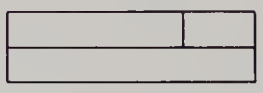
① □ + △ = 8 □ - △ = 2	② □ + △ = 15 □ - △ = 5
③ □ + △ = 20 □ - △ = 0	④ □ + △ = 9 □ - △ = 1

Follow the class work with prepared Ditto sheets, with four or more Sum and Difference games to a sheet, for individual exploration.

The Triple Game (Fact Teams)

A very useful and interesting game for developing the idea of fact teams is the Triple Game.

Have the pupils place a yellow rod and a red rod on their desks. Have them use their rods to construct a train and find the rod equal in length to that of the train, as shown below:



This triple, whose components are 5, 2, and 7, can yield information such as the following:

5 + 2 = 7 (reoding from left to right)

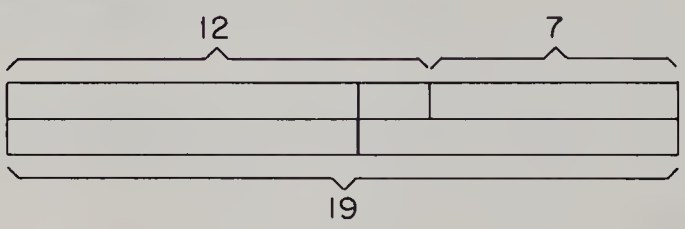
2 + 5 = 7 (reoding from right to left)

7 − 5 = 2 (the block rod covered by the yellow rod)

7 − 2 = 5 (the black rod covered by the red rod)

Although there are many more ways of writing about this number triple, it is important that your pupils learn at least four ways (the four facts of the team) of describing the triple. Write the fact team on the board; read the sentences to the class; have pupils read them to the rest of the class.

For Triple Games with second decade numbers, proceed somewhat as follows. Have pupils construct a train for 12 and another for 7. Write on the board: 12 + 7 = ____ Have them construct a train whose length equals that of the sum of 12 and 7 so that their triple (12, 7, 19) is represented by:



The fact team associated with this triple is:

12 + 7 = 19	19 − 7 = 12
7 + 12 = 19	19 − 12 = 7

Have a pupil suggest a triple for the class to construct, only this time instead of arranging the rods in a horizontal fashion have the pupils arrange the rods vertically. Suppose that a pupil suggested the triple (5, 6, 11) represented below:

The vertical form for the fact team can be written as follows:

5	11	6	11
+6	− 6	+5	− 5
11	5	11	6

Two-member teams can be organized to play Triple Games as follows:

A selects one length.

B selects another length.

A constructs the train.

B constructs the third member of the triple.

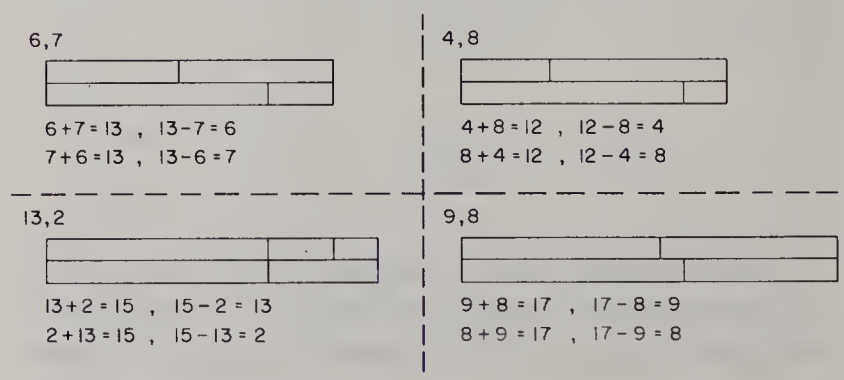
A and B both write the fact team for the triple and check each other's records.

The Triple Game can be individualized as follows:

Prepare Ditto sheets with four pairs of numbers listed on them:

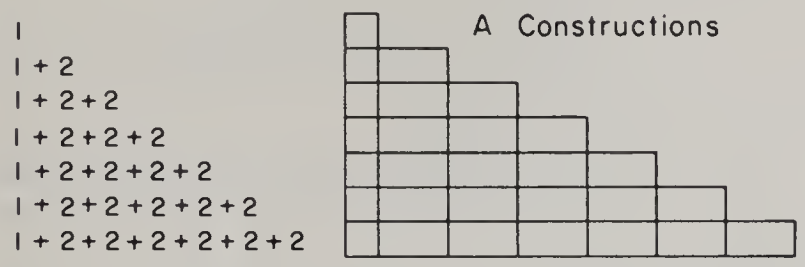
6,7	4,8
13,2	9,8

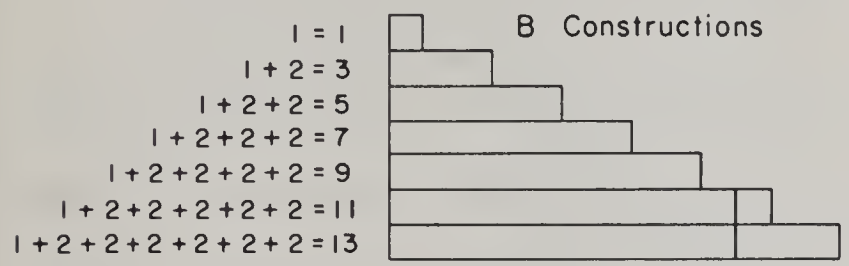
Pupils write the fact team associated with each pair by first using rods to form the appropriate triple, and then writing about the triple. Here is a picture of the Dittoed sheet showing the constructions and the fact teams:



The Staircase Game

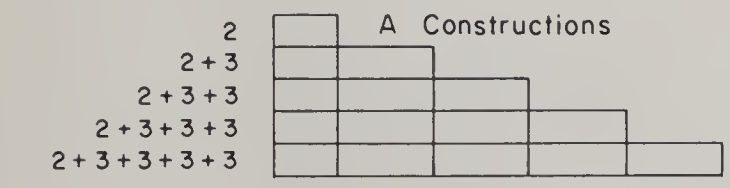
Form two-member teams. Ask each pupil to place a white rod on his desk. Each A member is to place a train of another white rod and a red rod under his white rod. Each B member is to place a rod as long as the white and red train under his white rod. Member A now places a W + R + R train under the W + R train. Member B measures this last A train and places a yellow rod under his light green rod. Continue for as many stages as is convenient. A sketch of seven stages looks like this:

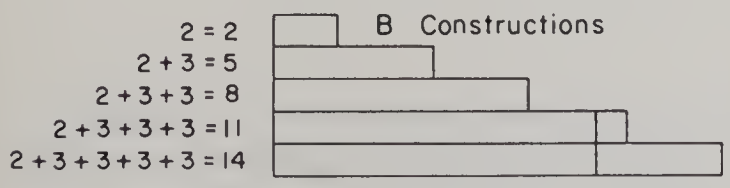


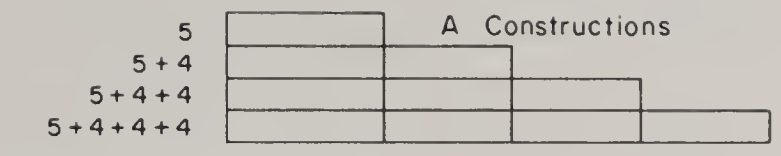


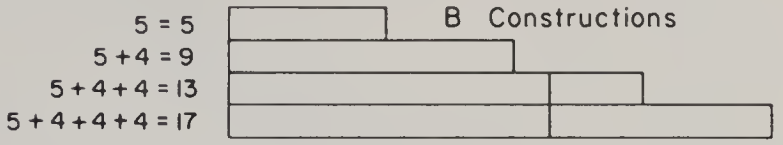
As each new extension is made, you and the class keep a record of the A and B results on the board.

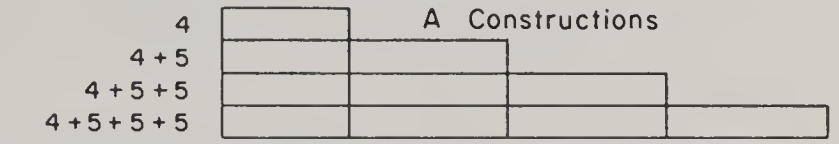
Some samples of other Staircase Games are shown below:

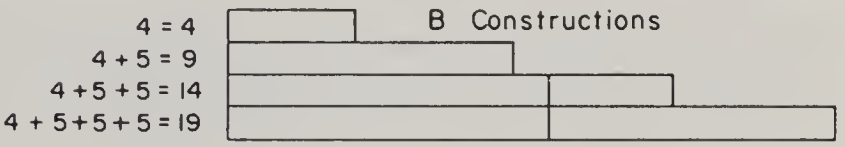








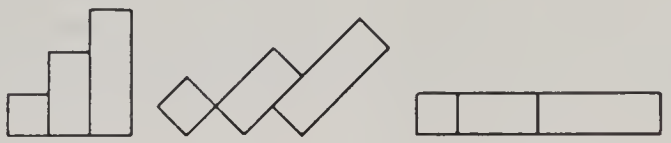




The Collapsing Staircase Game (Sequences and Series)

Have the class make a staircase with a white, a red, and a light green rod. Ask which rod is needed to climb from one step to the next. Have them use a white rod to climb the stairs.

Have the pupils collapse their staircase so as to form a train as shown below:



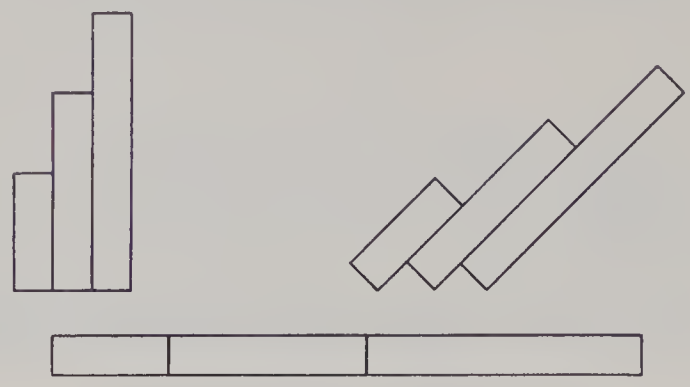
Ask what rod is as long as the train made by collapsing the staircase. Write the sentence on the board:

$$1 + 2 + 3 = 6$$

Have the class build a three-step staircase by starting with 3 and adding two each time until there are three rods in the staircase. Have them collapse the staircase to form a train. Ask what length is as long as this train. Write the sentence describing their actions on the board:

$$3 + 5 + 7 = 15$$

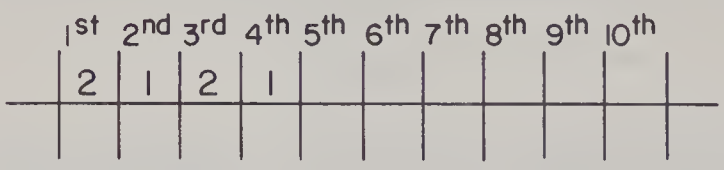
A sketch of their actions would look like this:



This activity is extendable as each new decade is explored.

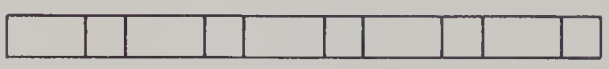
The Rhythm Game (Periodic Sequences)

Sketch the following on the board:

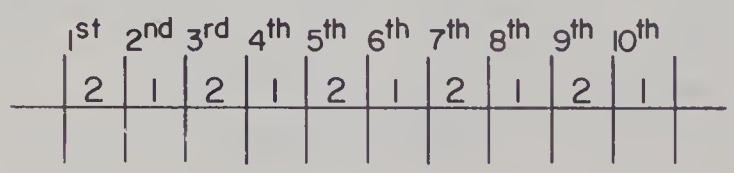


Have the pupils construct the train suggested by the entries in the table. Ask what the next car of the train would have to be if we wish to keep the rhythm going. Have them extend the train while you make your entry in the table. Have them add three more cars to the train so that the rhythm is maintained. Record the results in the table. Have them extend the pattern until there are ten cars in the train. Your record and their construction now looks like this:

Pupil constructions



Your record



Ask questions similar to the following (write the ordinal number names on the board):

"What's the *first* car in the train?"

"What's the *last* car in the train?"

"What's the *fourth* car in the train?"

"What's the *seventh* car in the train?"

"What's the *tenth* car in the train?"

Have the pupils stand each car up on its small face and slide the rods so they are adjacent to each other as shown below:

Ask the class to tell you what the up and down pattern is like. (Up two, down one, up two, down one . . .)

Have the class re-form the original train. Ask questions and guide the class as follows:

"Which rod is as long as the first car?"

"Place that rod on one side of your desk."

"Which rod is as long as the train of the first and second cars?"

"Place that rod next to the red rod on the side of your desk."

"Which rod is as long as the train of the first, second, and third cars?"

"Place it next to the red and light green rods on the side of your desk."

"Which rod is as long as the train of the first four cars?"

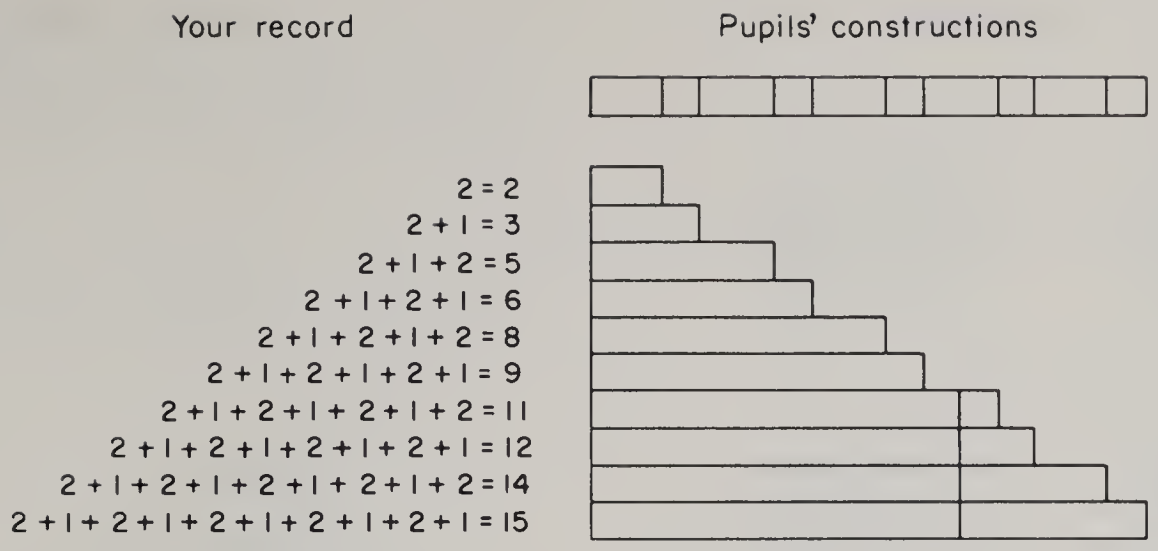
"Place it on the side with the other rods."

Continue until you complete the length of the train.

Write the following on the board:

2 = 2
2 + 1 = ___
2 + 1 + 2 = ___
2 + 1 + 2 + 1 = ___
2 + 1 + 2 + 1 + 2 = ___

Have the pupils read the answers by referring to the staircase on the side of their desks. A summary of the records and pupil desk arrangements looks like this:



Enter the sums in the original table as shown below:

1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
2	1	2	1	2	1	2	1	2	1
2	3	5	6	8	9	11	12	14	15

Ask questions similar to the following:

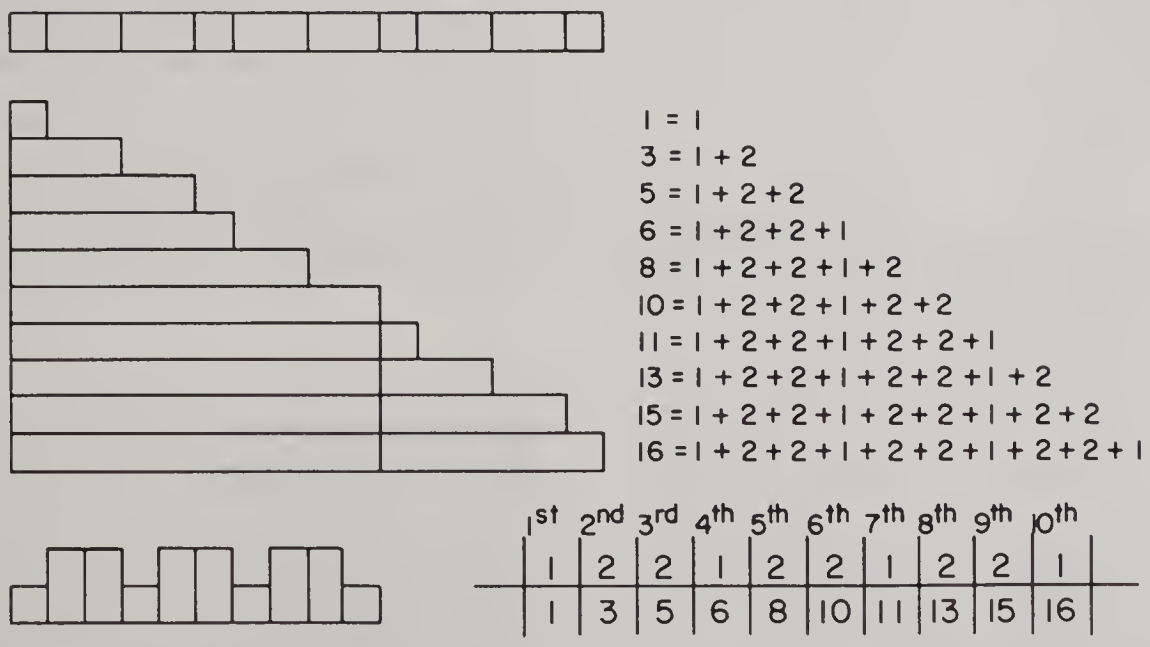
"What's the fifth car?" (Fifth term in the sequence?)

"What's the fifth sum?" (The sum of the first five terms of the sequence?)

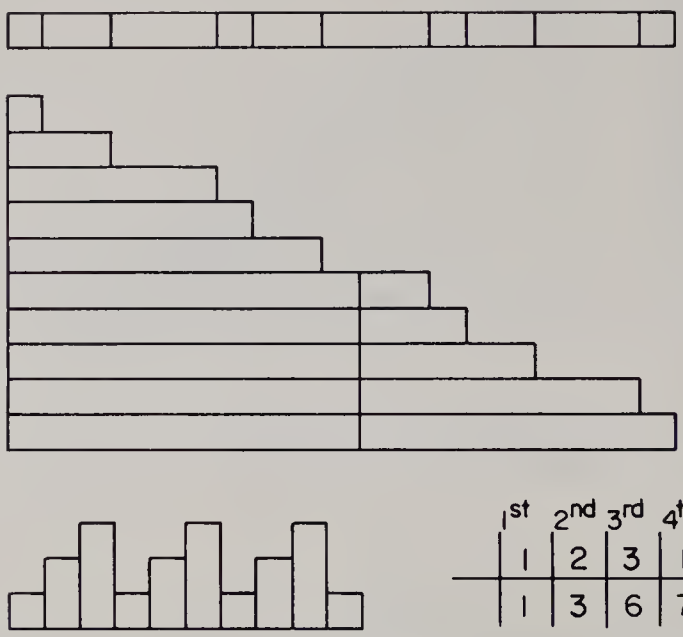
"What's the seventh sum?" (The sum of the first seven terms of the sequence?)

And so on.

Some samples of other sequences are as follows:



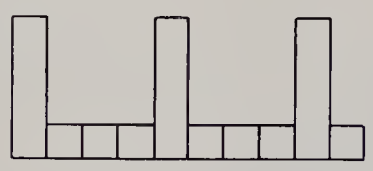
1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
1	2	2	1	2	2	1	2	2	1
1	3	5	6	8	10	11	13	15	16



1 = 1
3 = 1 + 2
6 = 1 + 2 + 3
7 = 1 + 2 + 3 + 1
9 = 1 + 2 + 3 + 1 + 2
12 = 1 + 2 + 3 + 1 + 2 + 3
13 = 1 + 2 + 3 + 1 + 2 + 3 + 1
15 = 1 + 2 + 3 + 1 + 2 + 3 + 1 + 2
18 = 1 + 2 + 3 + 1 + 2 + 3 + 1 + 2 + 3
19 = 1 + 2 + 3 + 1 + 2 + 3 + 1 + 2 + 3 + 1

1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
1	2	3	1	2	3	1	2	3	1
1	3	6	7	9	12	13	15	18	19

4 = 4
5 = 4 + 1
6 = 4 + 1 + 1
7 = 4 + 1 + 1 + 1
11 = 4 + 1 + 1 + 1 + 4
12 = 4 + 1 + 1 + 1 + 4 + 1
13 = 4 + 1 + 1 + 1 + 4 + 1 + 1
14 = 4 + 1 + 1 + 1 + 4 + 1 + 1 + 1
18 = 4 + 1 + 1 + 1 + 4 + 1 + 1 + 1 + 4
19 = 4 + 1 + 1 + 1 + 4 + 1 + 1 + 1 + 4 + 1

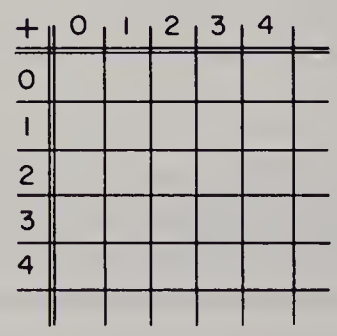


1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
4	1	1	1	4	1	1	1	4	1
4	5	6	7	11	12	13	14	18	19

The Row and Column Cell Building Game (Addition Tables)

Prepare oak tag sheets with centimeter grids on them, or use centimeter graph paper, so that each pupil has one for desk work.

Sketch the following on the board:

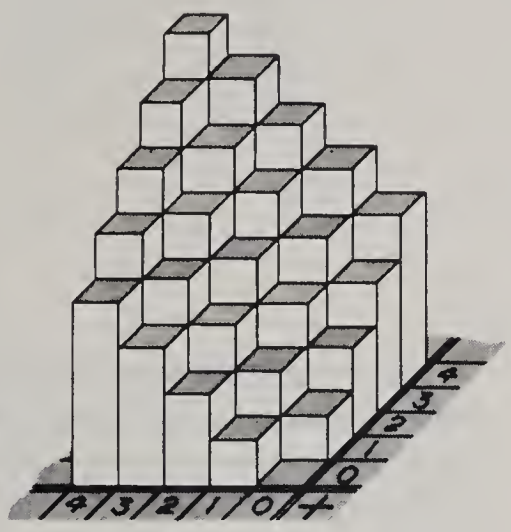


+	0	1	2	3	4
0					
1					
2					
3					
4					

Ask what rods they would stand up in the first row of cells to show the sums $0 + 0$, $0 + 1$, $0 + 2$, $0 + 3$, $0 + 4$. As each sum is determined, have pupils stand the appropriate rod on the appropriate cell.

Proceed to each successive row until each cell is occupied by a rod. A completed table and picture of the pupil construction would look like this:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	5
2	2	3	4	5	6
3	3	4	5	6	7
4	4	5	6	7	8



You can erase certain cell entries at random and ask the class to help you reconstruct the table, or you can have them write as many number sentences as they can of the form $\triangle + \bigcirc = \square$ where $\triangle$ holds the place for a row number, $\bigcirc$ holds the place for a column number, and $\square$ holds the place for a cell number.

The table can be used for subtraction situations by having pupils select a cell entry, read vertically up to the column label to find one of the addends, and horizontally to the row label for the other addend. Write on the board $\square - \bigcirc = \triangle$ and have pupils construct other number sentences for this pattern by using their tables.

Some variations on the table are as follows:

+	0	1	2	3	4
4	4	5	6	7	8
3	3	4	5	6	7
2	2	3	4	5	6
1	1	2	3	4	5
0	0	1	2	3	4

+	1	3	0	4	2
3	4	6	3	7	5
2	3	5	2	6	4
4	5	7	4	8	6
0	1	3	0	4	2
1	2	4	1	5	3

Still other possibilities for tables are as follows:

+	0	2	4	6
0	0	2	4	6
2	2	4	6	8
4	4	6	8	10
6	6	8	10	12

+	1	3	5
1	2	4	6
3	4	6	8
5	6	8	10

+	0	2	4	6
1	1	3	5	7
3	3	5	7	9
5	5	7	9	11

+	2	3	5
2	4	5	7
3	5	6	8
5	7	8	10

Written tabular arrays can be Dittoed so that pupils can explore patterns which take them well into the second decade numbers. Pupils won't need to construct rod models for these longer tables unless they wish to. Some possibilities for tables are the following:

+	0	1	2	3	4	5	6	7	8	9	10
0											
1											
2											
3											
4											
5											
6											
7											
8											
9											
10											

+	0	2	4	6	8	10
0						
2						
4						
6						
8						
10						

+	1	3	5	7	9
1					
3					
5					
7					
9					

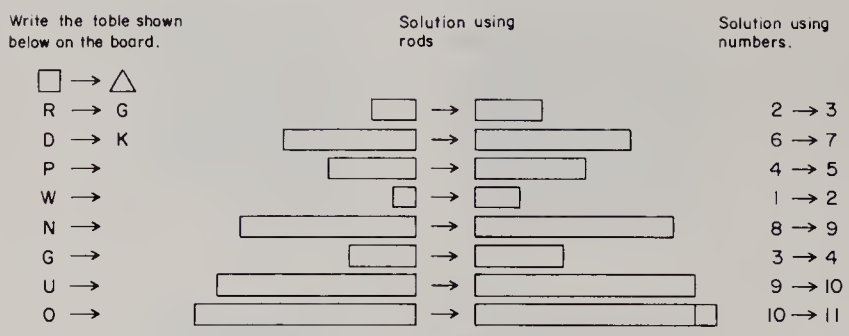
+	0	2	4	6	8	10
1						
3						
5						
7						
9						

+	2	3	5	7
2				
3				
5				
7				

+	1	3	6	10
1				
3				
6				
10				

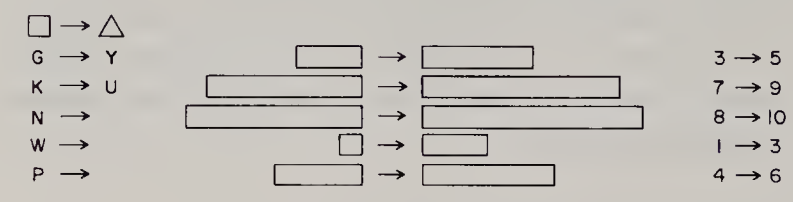
+	0	1	4	9
0				
1				
4				
9				

The What's-My-Rule? Game



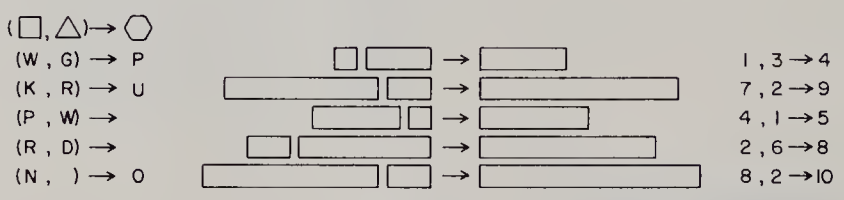
Ask the class to construct with rods the table you have on the board and then tell you what rod P "goes to." If no one suggests Y, you write it on the board. Have the class work out the pattern at their desks before you complete it on the board.

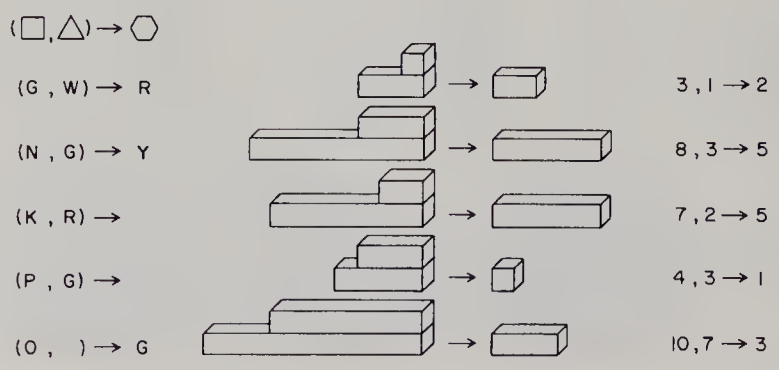
Start another game, such as:



Have the class construct and complete the pattern before you discuss the rule with them.

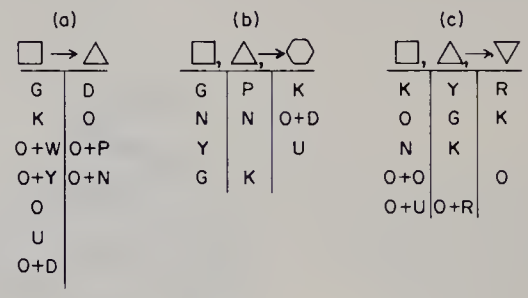
Two ways to present What's-My-Rule? Games involving pairs of numbers are shown in the following pictures:





Each of the completed patterns can be translated into number patterns (as shown in the third column) by using the appropriate numbers represented by the rods.

Some other What's My Rule? games which will take pupils into the second decade are:

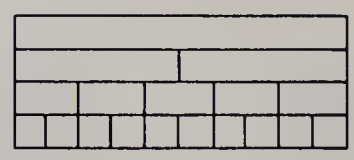


Have pupils form two-member teams to make up their own What's-My-Rule? games. Eventually they should use the number names for the rods. Encourage pupils to invent rules they think will stump the class. Have them post their games on the bulletin board and leave the rule with you. Pupils who find the secret can demonstrate their discovery by extending the pattern.

CUISENAIRE® RODS—SECTION IV: MULTIPLICATION AND DIVISION—FIRST DECADE

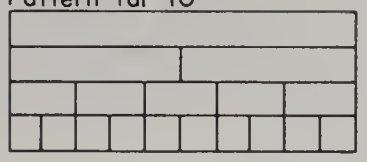
Product Patterns

Have each pupil place an orange rod on his desk. Suggest to the class that they try to construct a train of rods of one color which is as long as the orange rod. Discuss the constructions with the class. Each construction offered by a pupil should be tested by the others so that eventually each pupil has the following pattern on his desk:

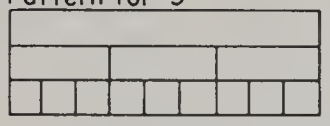


Proceed to the blue rod and to each of the other rods until the class has constructed each of the patterns shown in the next column:

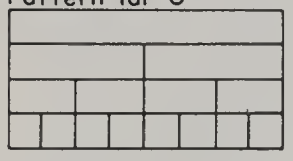
Pattern for 10



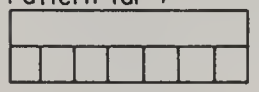
Pattern for 9



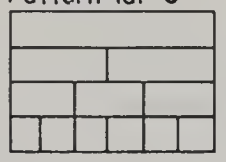
Pattern for 8



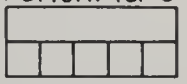
Pattern for 7



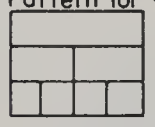
Pattern for 6



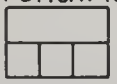
Pattern for 5



Pattern for 4



Pattern for 3



Pattern for 2

Pattern for 1

Have the class reconstruct the pattern for 10. Ask:

"How many five rods are in the pattern?"

"How many two rods are in the pattern?"

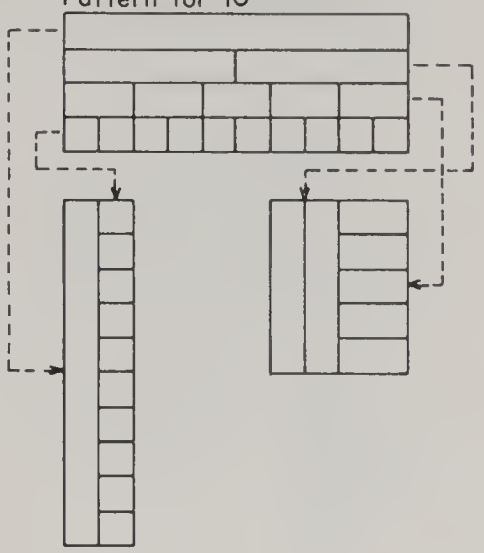
"How many one rods are in the pattern?"

"How many ten rods are in the pattern?"

Have the class remove the train of yellows from the pattern and arrange the cars so that instead of being end to end, they are now side by side. Ask them to do the same thing with the train of reds. Have them discuss the two floors thus obtained. They should notice that the floor of two fives fits on the floor of five twos, and vice-versa.

Have them arrange the floors obtained by rearranging the trains so that floors that fit are placed adjacent to each other as shown in the next column:

Pattern for 10



(Of course, no rearrangement of trains is required for the orange and white floors.)

Develop a discussion about their floors so that questions like the following are asked:

"How many tens in the orange floor?"

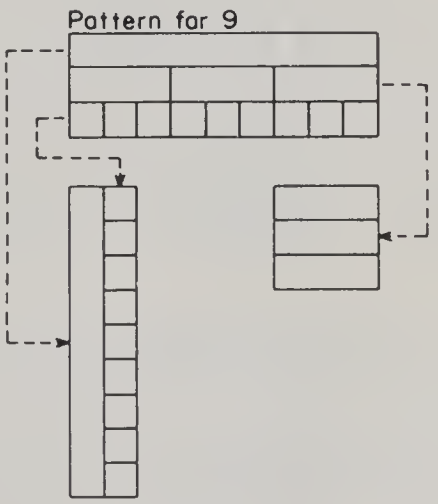
"How many ones in the white floor?"

"How many fives in the yellow floor?"

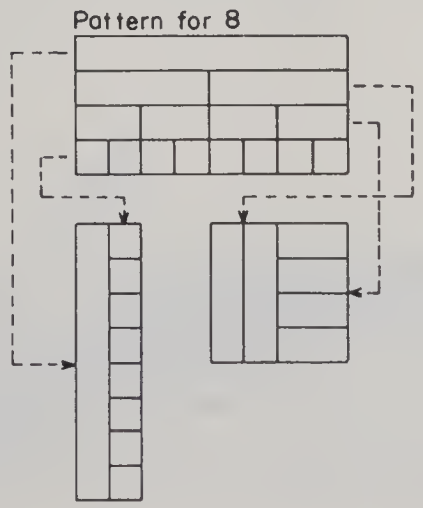
"How many twos in the red floor?"

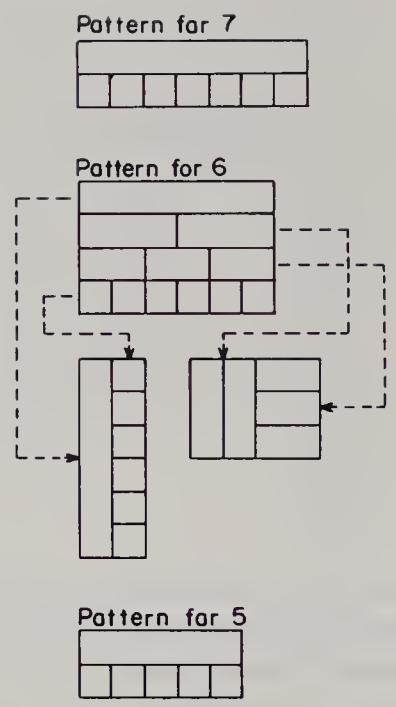
Have the class construct floors by rearranging the trains for each of the other rods so that eventually they will have constructed and discussed each of the patterns sketched below:

Pattern for 9



Pattern for 8





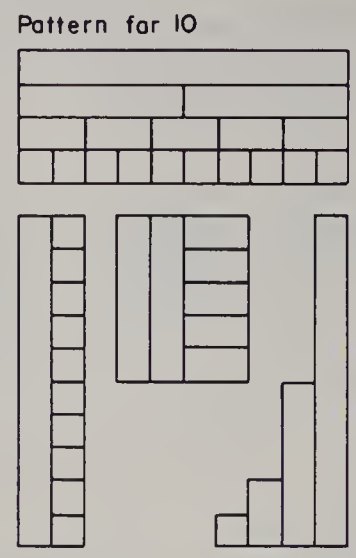
Factor Sets

Have the class reconstruct the train and floor patterns for 10. Ask the class the following:

"Which rods did you use to make the train pattern for 10?" (white, red, yellow, orange)

"Which rods did you use to make the floor pattern for 10?" (white, red, yellow, orange)

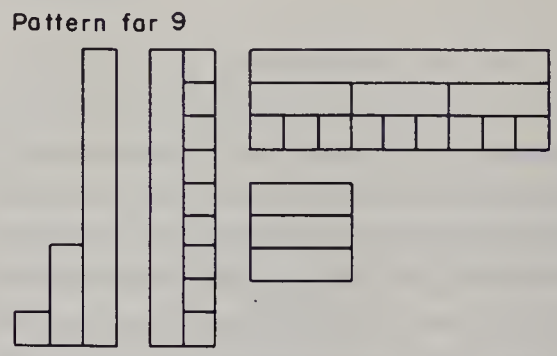
Have them place on their desks a set of rods to show which rods were used. Their constructions should now look like this:



Tell the class that the numbers 1, 2, 5, 10 are called the *factors* of 10. Start a record on the board similar to the following:

Number	Members of the set of factors
10	1,2,5,10

Ask for the set of factors for 9. Have the pupils prove they have the correct set of factors by displaying the floor and train patterns to go with the factor set. Their display would look like this:

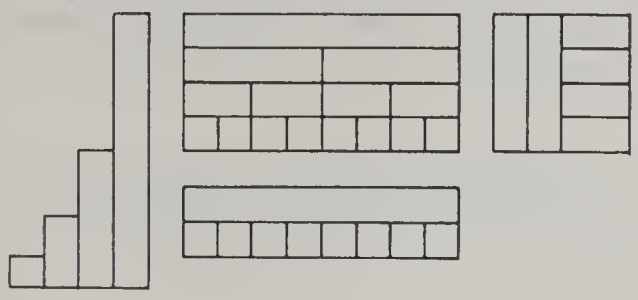


After the class completes the constructions, enter the set of factors in the table on the board. The record is extended as follows:

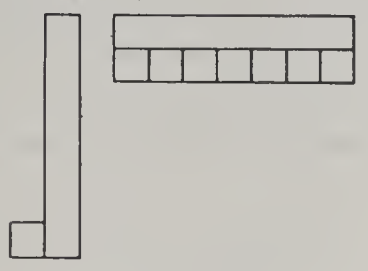
Number	Members of the set of factors
10	1,2,5,10
9	1,3,9

Have the class proceed in a similar way to explore the sets of factors for each of the other numbers from 1 through 10 so that eventually all factor sets have been constructed, verified, and recorded. Below is a summary of their actions for the numbers 1-8:

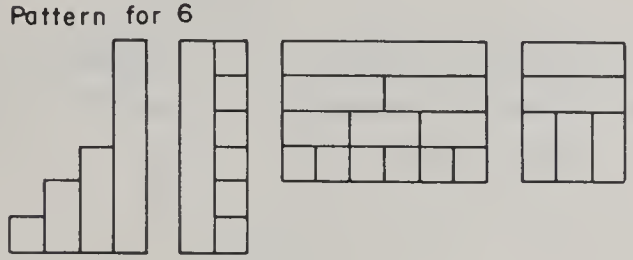
Pattern for 8



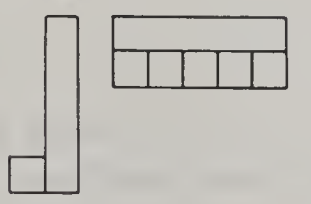
Pattern for 7



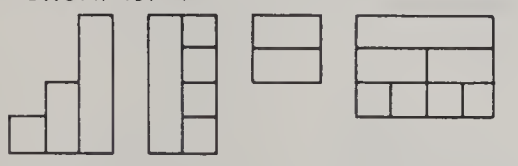
Pattern for 6



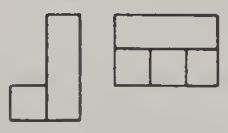
Pattern for 5



Pattern for 4



Pattern for 3



Pattern for 2

Pattern for 1

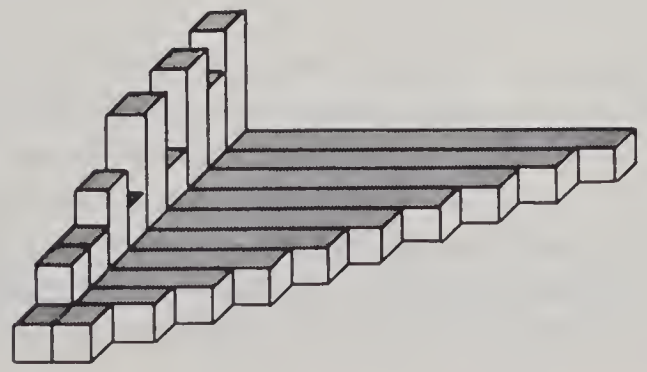
Number	Members of the set of factors
1	1
2	1 , 2
3	1 , 3
4	1 , 2 , 4
5	1 , 5
6	1 , 2 , 3 , 6
7	1 , 7
8	1 , 2 , 4 , 8
9	1 , 3 , 9
10	1 , 2 , 5 , 10

Extend your table to include information about the number of factors for each number. Your record should look like this:

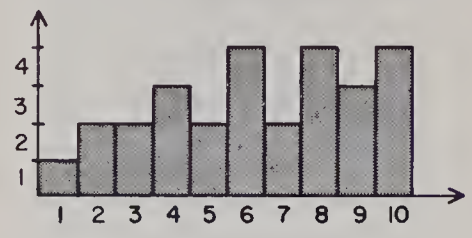
Number	Members of the set of factors	Number of factors
1	1	1
2	1 , 2	2
3	1 , 3	2
4	1 , 2 , 4	3
5	1 , 5	2
6	1 , 2 , 3 , 6	4
7	1 , 7	2
8	1 , 2 , 4 , 8	4
9	1 , 3 , 9	3
10	1 , 2 , 5 , 10	4

Products, Factors, and Factorizations

Have each pupil build a staircase from W to O, flat on the desk. Have him place next to each rod in the staircase a rod which represents the number of factors for the staircase rod. His display will look like this:



Have pupils help you to construct a bar graph depicting each number and its number of factors.



Ask questions similar to the following:

"Which of the numbers from 1 through 10 have just one factor?"

"Which have just two factors?"

"Which have more than two factors?"

Have the class assist you in constructing the following table:

Just one factor	Just two factors	More than two factors
1	2	4
	3	6
	5	8
	7	9
		10

Tell the class that numbers with just two factors are called *prime numbers*, numbers with more than two factors are called *composite numbers*, and the number *one* is in a special class by itself since it has just one factor.

Prepare a Ditto sheet with the following information on it:

Number	Members of the set of factors	Number of factors	Prime	Composite	Special
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					

Form four-member play teams.

A selects a number from 1-10 and constructs the train pattern for the number.

B constructs the corresponding floor pattern.

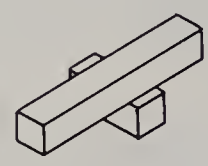
C constructs the factor set.

D consults with A, B, and C, and records the information in the chart for the number selected by A.

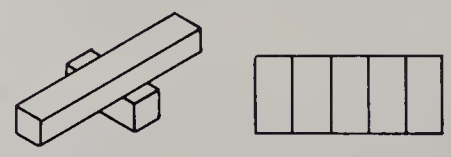
Each team member should have experience at being the number selector and train maker, the floor maker, the factor set maker, and the recorder. Continue the activity until each team has completed its chart. The class can now help you complete your chart on the board so that it eventually looks like this:

Number	Members of the set of factors	Number of factors	Prime	Composite	Special
1	1	1			✓
2	1, 2	2	✓		
3	1, 3	2	✓		
4	1, 2, 4	3		✓	
5	1, 5	2	✓		
6	1, 2, 3, 6	4		✓	
7	1, 7	2	✓		
8	1, 2, 4, 8	4		✓	
9	1, 3, 9	3		✓	
10	1, 2, 5, 10	4		✓	

Have the class form the train pattern for 10. Form a cross by crossing a red rod by a yellow rod as shown below:



Tell the pupils the cross is a signal for them to construct a floor. Have them guess which one it is. The symbol should suggest a floor of red rods as long as a yellow rod, as sketched below:



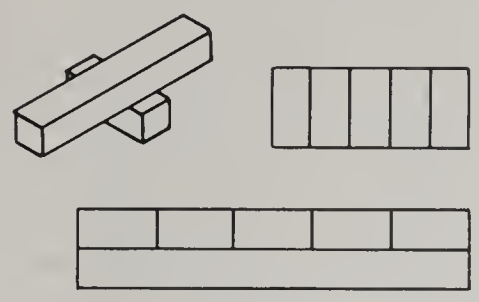
Have them rearrange the floor into a train and find the rod whose length is the length of the train. Write the following on the board:

$$5 \times 2 = 10$$

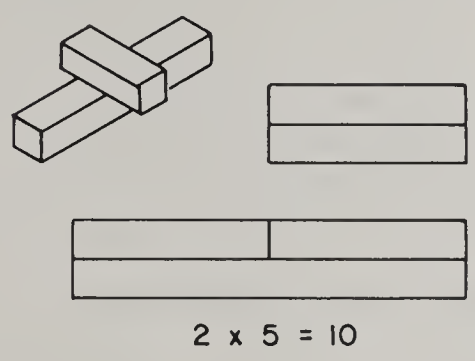
Ask if anyone knows how to read the sentence. If no one offers a correct version, tell them it is read:

Five times two equals ten

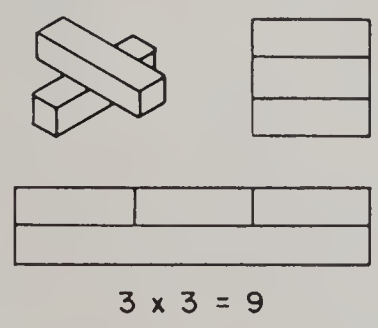
Have the pupils summarize the sequence of actions by constructing and writing about it as follows:



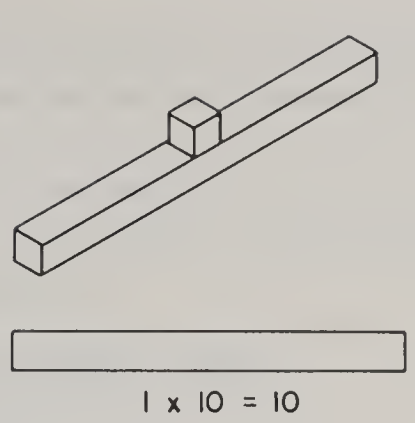
Change the order of the rods in the cross and have the class construct the floor and train pattern suggested by the new symbol. A sketch of their new constructions looks like this:

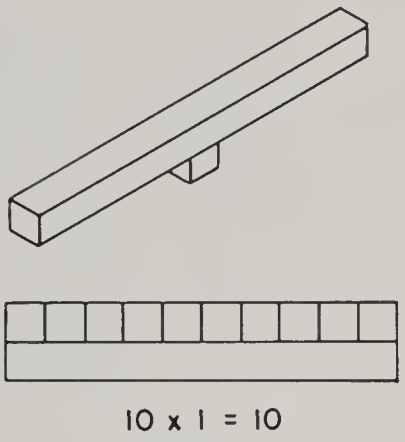


Form a cross using two light green rods. Have the class construct the floor and train patterns suggested by this symbol. A sketch of their constructions would look like this:

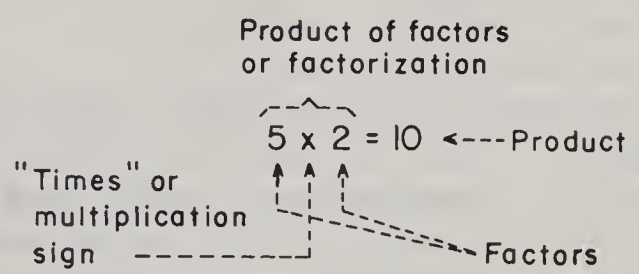


Special crosses to symbolize 1 x 10 and 10 x 1 and the corresponding floor and train patterns are sketched in the next column.

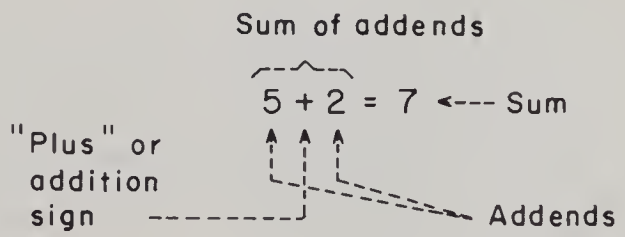




Before proceeding with pupil teams, you might want to introduce the class to the terminology associated with a number sentence which describes a multiplication situation.



Contrast the above with an addition situation.



Although these terms are not nearly as important as the concepts they name, they are useful for purposes of discussion.

Your pupils will tend to want to use such terms so long as they are suggestive of actions they already know how to perform. To guard against your shifting the focus from concepts to terms, use the

terms only after you are sure that your pupils thoroughly understand the concepts named by the terms.

Display a copy of the following chart on the board, and prepare Ditto copies of it:

Number	factors	factorizations	No. of factors	Prime	Composite	Special	Number sentences
1							
2							
3							
4							
5							
6							
7							
8							
9							
10	1, 2, 5, 10	1 x 10, 10 x 1, 2 x 5, 5 x 2	4		✓		1 x 10 = 10, 10 x 1 = 10 2 x 5 = 10, 5 x 2 = 10

With the assistance of the class, complete the row for the number 10, as shown above.

Have the class form five-member work teams so that:

A selects a number and constructs the multiplication train pattern.

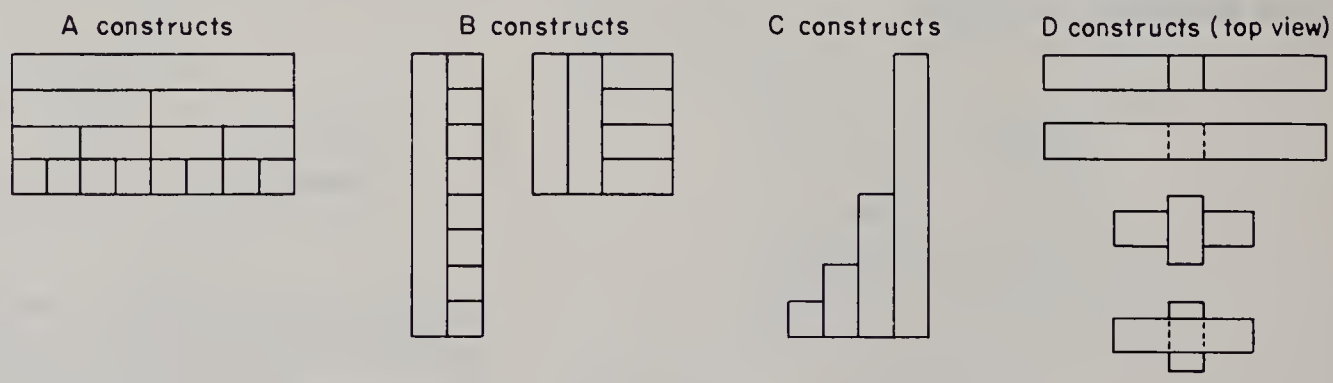
B constructs the floor pattern.

C constructs the factor set.

D constructs the factorization set (set of crosses).

E records the team's findings on the chart.

A sketch of a team's activities for the number 8 looks like the following:



E records

Number	factors	factorizations	No. of factors	Prime	Composite	Special	Number sentences
8	1, 2, 4, 8	1 x 8, 8 x 1 4 x 2, 2 x 4	4		✓		1 x 8 = 8, 8 x 1 = 8 2 x 4 = 8, 4 x 2 = 8

Each team should explore each of the numbers from 1 through 10 and complete its record. The completed chart will look like this:

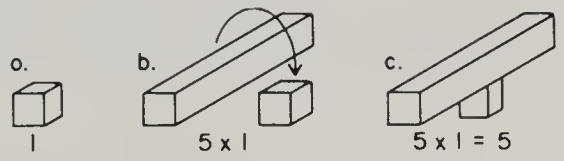
Number	factors	factorizations	No. of factors	Prime	Composite	Special	Number sentences
1	1	1 x 1	1			✓	1 x 1 = 1
2	1, 2	1 x 2, 2 x 1	2	✓			1 x 2 = 2, 2 x 1 = 2
3	1, 3	1 x 3, 3 x 1	2	✓			1 x 3 = 3, 3 x 1 = 3
4	1, 2, 4	1 x 4, 4 x 1, 2x2	3		✓		1 x 4 = 4, 4 x 1 = 4, 2 x 2 = 4
5	1, 5	1 x 5, 5 x 1	2	✓			1 x 5 = 5, 5 x 1 = 5
6	1, 2, 3, 6	1 x 6, 6 x 1, 2 x 3, 3 x 2	4		✓		1 x 6 = 6, 6 x 1 = 6 2 x 3 = 6, 3 x 2 = 6
7	1, 7	1 x 7, 7 x 1	2	✓			1 x 7 = 7, 7 x 1 = 7
8	1, 2, 4, 8	1 x 8, 8 x 1, 2 x 4, 4 x 2	4		✓		1 x 8 = 8, 8 x 1 = 8, 2 x 4 = 8, 4 x 2 = 8
9	1, 3, 9	1 x 9, 9 x 1, 3x3	3		✓		1 x 9 = 9, 9 x 1 = 9, 3 x 3 = 9
10	1, 2, 5, 10	1 x 10, 10 x 1 2 x 5, 5 x 2	4		✓		1 x 10 = 10, 10 x 1 = 10, 2 x 5 = 10, 5 x 2 = 10

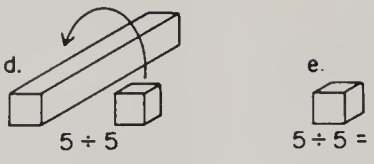
Division—the "Undoing" of Multiplication

Have pupils place a white rod on their desks. Ask them to cross it with a yellow rod. $5 \times 1 = 5$ describes this situation. Lift the yellow rod from the white rod and write $5 \div 5 =$ ____. Tell the class that the $\times$ represents the making of the cross whereas $\div$ represents the unmaking, or undoing, of the cross. Ask what rod is still on their desk. Complete the sentence and say it aloud:

$5 \div 5 = 1$ Five divided by five equals one.

The sketch in the next column pictures the situations just described.





Have two-member teams make and unmake crosses so that eventually the record below represents their actions and their spoken words related to those actions.

	Factor X Factor = Product		Product ÷ Factor = Factor	
If	10 X 1 = 10	then	10 ÷ 10 = 1	If ten times one equals ten then ten divided by ten equals one.
If	1 X 10 = 10	then	10 ÷ 1 = 10	If one times ten equals ten then ten divided by one equals ten.
If	9 X 1 = 9	then	9 ÷ 9 = 1	If nine times one equals nine then nine divided by nine equals one.
If	1 X 9 = 9	then	9 ÷ 1 = 9	If one times nine equals nine then nine divided by one equals nine.
If	8 X 1 = 8	then	8 ÷ 8 = 1	If eight times one equals eight then eight divided by eight equals one.
If	1 X 8 = 8	then	8 ÷ 1 = 8	If one times eight equals eight then eight divided by one equals eight.
If	7 X 1 = 7	then	7 ÷ 7 = 1	If seven times one equals seven then seven divided by seven equals one.
⋮	⋮	⋮	⋮	⋮
If	2 X 1 = 2	then	2 ÷ 2 = 1	If two times one equals two then two divided by two equals one.
If	1 X 2 = 2	then	2 ÷ 1 = 2	If one times two equals two then two divided by one equals two.
If	1 X 1 = 1	then	1 ÷ 1 = 1	If one times one equals one then one divided by one equals one.

Have pupils place a white rod on their desks. Ask them to cross it with a yellow rod and then cross that with a red rod. Ask for a number sentence to describe this set of actions. If no one suggests the appropriate one, write on the board the following:

$$1 = 1$$
$$5 \times 1 = 5$$
$$2 \times 5 \times 1 = 10$$

Have the class undo the multiplication by two and ask if this number sentence describes their actions:

$$10 \div 2 = 5$$

Now have them undo the multiplication by 5 and ask if $5 \div 5 = 1$ describes their actions.

Have them reconstruct the original "tower":

$$2 \times 5 \times 1 = 10$$

Tell them to divide 10 by 5. Ask if $10 \div 5 = 2$ describes their actions.

Now have them divide 2 by 2. Ask if $2 \div 2 = 1$ describes their actions.

Write the following on the board:

If $2 \times 3 \times 1 =$ ___
If $4 \times 2 \times 1 =$ ___
If $2 \times 4 \times 1 =$ ___
If $3 \times 2 \times 1 =$ ___
If $7 \times 1 \times 1 =$ ___
If $1 \times 8 \times 1 =$ ___
If $3 \times 3 \times 1 =$ ___
If $10 \times 1 \times 1 =$ ___
If $1 \times 5 \times 1 =$ ___

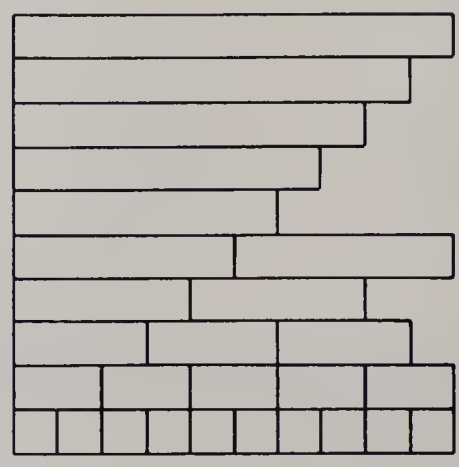
Guide the class through the construction of each tower to determine the corresponding product. Follow up each construction by the undoing of the tower, having pupils assist you in writing the appropriate division sentences. A completed table of the above activities looks like this:

If $2 \times 3 \times 1 = 6$, then $6 \div 2 = 3$ and $3 \div 3 = 1$
If $4 \times 2 \times 1 = 8$, then $8 \div 4 = 2$ and $2 \div 2 = 1$
If $2 \times 4 \times 1 = 8$, then $8 \div 2 = 4$ and $4 \div 4 = 1$
If $3 \times 2 \times 1 = 6$, then $6 \div 3 = 2$ and $2 \div 2 = 1$
If $7 \times 1 \times 1 = 7$, then $7 \div 7 = 1$ and $1 \div 1 = 1$
If $1 \times 8 \times 1 = 8$, then $8 \div 1 = 8$ and $8 \div 8 = 1$
If $3 \times 3 \times 1 = 9$, then $9 \div 3 = 3$ and $3 \div 3 = 1$
If $10 \times 1 \times 1 = 10$, then $10 \div 10 = 1$ and $1 \div 1 = 1$
If $1 \times 5 \times 1 = 5$, then $5 \div 1 = 5$ and $5 \div 5 = 1$

Have pupils take especial note of the fact that in constructing their product tower, they always started with a white rod and that when they undid their constructions (or divided), they always finished with the same white rod. Since this is so, they need not place the white rod on the desk to start with, but rather may assume its presence in any future towers they construct. Furthermore, in their undoing of future towers, they may also assume the presence of the white rod as the first rod in the tower. They will find this agreement a practical one, particularly when they construct larger and higher towers. Their towers will be far more stable when they don't have to build them upon a white rod.

Division—Repeated Subtraction

Have each pupil place an orange rod on his desk. Ask pupils to construct a train of blue rods as long as the orange rod. Many will note that one blue rod is not enough and two or more blue rods are too many. Have them try a brown rod. Similar observations will be made. Have them use each rod in succession to attempt to construct a train as long as the orange rod. A sketch of their efforts looks like this:



Place the following chart on the board:

Dividend	quotient	divisor	remainder	D = q x d + r	$d \overline{)D}^{q,r}$
10					
10	1	9	1	10 = (1x9) + 1	$9\overline{)10}^{1}$ rem 1
10	1	8	2	10 = (1x8) + 2	$8\overline{)10}^{1}$ rem. 2
10					
10					
10					
10					
10					
10					
10					

Ask the following questions and make the appropriate entries in the chart as pupils offer them:

"How many nines did you use to make your ten train?

"How much of the ten train was left over?"

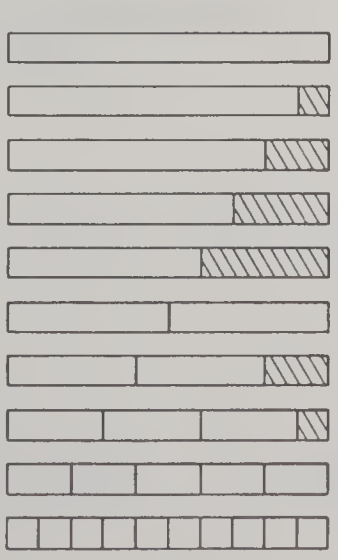
Complete the train by filling in the remainder. Show them the record in the table that describes their actions and discussion.

"How many eights did you use to make your ten train?"

"How much of the ten train remained left over?"

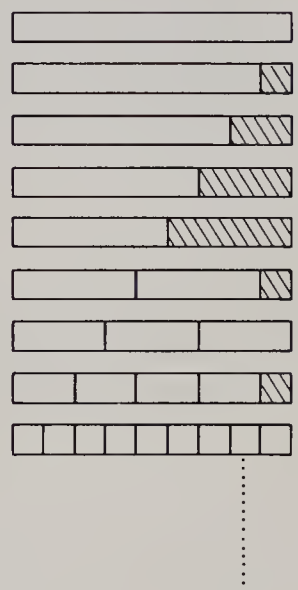
Complete the train by filling in the remainder rod. Show the class the record in the table that describes their actions and discussion.

Proceed with the class until all rows in the table are considered, completing each row not already complete and recorded in the table. A final version of the constructions and accompanying table would look like the following:



Dividend	quotient	divisor	remainder	D = q x d + r	$d \overline{)D}^{q,r}$
10	1	10	0	10 = (1 x 10) + 0	$10\overline{)10}^{1}$ rem. 0
10	1	9	1	10 = (1 x 9) + 1	$9\overline{)10}^{1}$ rem. 1
10	1	8	2	10 = (1 x 8) + 2	$8\overline{)10}^{1}$ rem. 2
10	1	7	3	10 = (1 x 7) + 3	$7\overline{)10}^{1}$ rem. 3
10	1	6	4	10 = (1 x 6) + 4	$6\overline{)10}^{1}$ rem. 4
10	2	5	0	10 = (2 x 5) + 0	$5\overline{)10}^{2}$ rem. 0
10	2	4	2	10 = (2 x 4) + 2	$4\overline{)10}^{2}$ rem. 2
10	3	3	1	10 = (3 x 3) + 1	$3\overline{)10}^{3}$ rem. 1
10	5	2	0	10 = (5 x 2) + 0	$2\overline{)10}^{5}$ rem. 0
10	10	1	0	10 = (10 x 1) + 0	$1\overline{)10}^{10}$ rem. 0

Have two-member teams construct division patterns for each of the numbers, 9 through 1. After each pattern is constructed, have all teams assist you in making and completing the record for their patterns. A summary of their actions and your records shows the following:



Dividend	quotient	divisor	remainder	D = q x d + r	$d\overline{)D}^{q,r}$
9	1	9	0	9 = (1 x 9) + 0	$9\overline{)9}^{1}$ rem. 0
9	1	8	1	9 = (1 x 8) + 1	$8\overline{)9}^{1}$ rem. 1
9	1	7	2	9 = (1 x 7) + 2	$7\overline{)9}^{1}$ rem. 2
9	1	6	3	9 = (1 x 6) + 3	$6\overline{)9}^{1}$ rem. 3
9	1	5	4	9 = (1 x 5) + 4	$5\overline{)9}^{1}$ rem. 4
9	2	4	1	9 = (2 x 4) + 1	$4\overline{)9}^{2}$ rem. 1
9	3	3	0	9 = (3 x 3) + 0	$3\overline{)9}^{3}$ rem. 0
9	4	2	1	9 = (4 x 2) + 1	$2\overline{)9}^{4}$ rem. 1
9	9	1	0	9 = (9 x 1) + 0	$1\overline{)9}^{9}$ rem. 0

Dividend	quotient	divisor	remainder	D = q x d + r	$d\overline{)D}^{q,r}$
3	1	3	0	3 = (1 x 3) + 0	$3\overline{)3}^{1}$ rem. 0
3	1	2	1	3 = (1 x 2) + 1	$2\overline{)3}^{1}$ rem. 1
3	3	1	0	3 = (3 x 1) + 0	$1\overline{)3}^{3}$ rem. 0

Dividend	quotient	divisor	remainder	D = q x d + r	$d\overline{)D}^{q,r}$
2	1	2	0	2 = (1 x 2) + 0	$2\overline{)2}^{1}$ rem 0
2	2	1	0	2 = (2 x 1) + 0	$1\overline{)2}^{2}$ rem. 0

Dividend	quotient	divisor	remainder	D = q x d + r	$d\overline{)D}^{q,r}$
1	1	1	0	1 = (1 x 1) + 0	$1\overline{)1}^{1}$ rem. 0

Random Sampling of Division Facts

Write the following on the board:

Dividend = quotient x divisor + remainder

10 = (2 x 5) + 0

$$\text{divisor}\ \overline{)\text{Dividend}}^{\text{quotient, remainder}}$$

Have pupils tell you how to complete the entry on the right. Since 2 is the quotient, 5 is the divisor, and 0 is the remainder, a correct completion is:

$$5 \overline{)\ 10}\quad 2,\ \text{rem. } 0$$

But, since multiplication is commutative, a second correct completion based on the fact that $10 = (2 \times 5) + 0$ is:

$$2 \overline{)\ 10}\quad 5,\ \text{rem. } 0$$

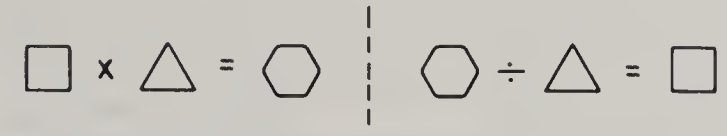
Have pupils tell you how to complete the right-hand entry in the following:

$9 = (2 \times 4) + 1$ $\qquad\qquad$ $\overline{)\ \ \ }$

Prepare Ditto sheets similar to the one shown below for two-member teams to work on. Pupils should check each other's work, using the rods to settle any disagreements.

Relating Multiplication and Division Sentences

Write the following on the board:



List the 27 sentences about mulitplication for the numbers from 1 through 10 in the left-hand column. Have pupils use their rods to form crosses representing each product. Then have them undo

Dividend = quotient x divisor + remainder	divisor $\overline{)\ \text{Dividend}}$ quotient, remainder
$7 = 3 \times 2 + 1$	
$9 = 1 \times 5 + 4$	
$4 = 1 \times 3 + 1$	
$6 = 1 \times 4 + 2$	
$10 = 1 \times 7 + 3$	

etc.

After pupils finish their worksheets, discuss their work and record the information on the board. Prepare other Ditto sheets in which the pattern is the reverse of the one shown above:

divisor $\overline{)\ \text{Dividend}}$ quotient, remainder	Dividend = quotient x divisor + remainder
$2 \overline{)\ 7}\quad 3$ rem. 1	
$7 \overline{)\ 9}\quad 1$ rem. 2	
$4 \overline{)\ 10}\quad 2$ rem. 2	
$3 \overline{)\ 5}\quad 1$ rem. 2	

etc.

each cross and record the division sentence that corresponds to it so that the final chart looks like this:

□ x △ = ⬡	⬡ ÷ △ = □
1 x 1 = 1	1 ÷ 1 = 1
2 x 1 = 2	2 ÷ 1 = 2
1 x 2 = 2	2 ÷ 2 = 1
3 x 1 = 3	3 ÷ 1 = 3
1 x 3 = 3	3 ÷ 3 = 1
4 x 1 = 4	4 ÷ 1 = 4
1 x 4 = 4	4 ÷ 4 = 1
2 x 2 = 4	4 ÷ 2 = 2
⋮	⋮
10 x 1 = 10	10 ÷ 1 = 10
1 x 10 = 10	10 ÷ 10 = 1
5 x 2 = 10	10 ÷ 2 = 5
2 x 5 = 10	10 ÷ 5 = 2

Multiplication and Division Fact Teams

Have each pupil place a yellow rod and a red rod on his desk. Tell the class there is a fact team of 4 sentences which describe the 4 situations they would have if they crossed and uncrossed each rod by the other. Ask if anyone knows the fact team. What you are after, of course, is:

2 x 5 = 10 10 ÷ 5 = 2
5 x 2 = 10 10 ÷ 2 = 5

Have each pupil place a white rod and an orange rod on his desk and ask for the fact team that would describe the making and unmaking of crosses with these rods. Record their findings.

1 x 10 = 10 10 ÷ 10 = 1
10 x 1 = 10 10 ÷ 1 = 10

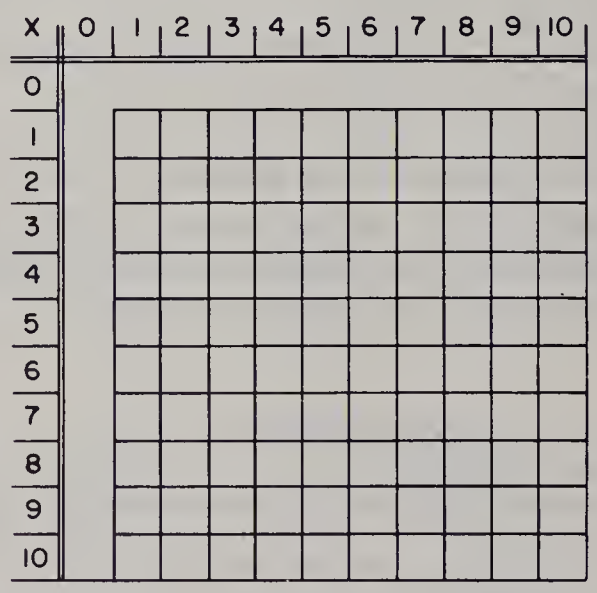
Proceed in a similar fashion until all the fact teams listed below are considered:

Number Pairs	Fact Team	
(1, 1)	1 x 1 = 1	1 ÷ 1 = 1
(1, 2)	1 x 2 = 2 2 x 1 = 2	2 ÷ 2 = 1 2 ÷ 1 = 2
(1, 3)	1 x 3 = 3 3 x 1 = 3	3 ÷ 3 = 1 3 ÷ 1 = 3
⋮	⋮	⋮
(1, 10)	1 x 10 = 10 10 x 1 = 10	10 ÷ 10 = 1 10 ÷ 1 = 10
(2, 2)	2 x 2 = 4	4 ÷ 2 = 2
(2, 3)	2 x 3 = 6 3 x 2 = 6	6 ÷ 3 = 2 6 ÷ 2 = 3
(2, 4)	2 x 4 = 8 4 x 2 = 8	8 ÷ 4 = 2 8 ÷ 2 = 4
(2, 5)	2 x 5 = 10 5 x 2 = 10	10 ÷ 5 = 2 10 ÷ 2 = 5
(3, 3)	3 x 3 = 9	9 ÷ 3 = 3

Two-member teams can be formed to play the Fact Team game. A selects a pair of rods, and B names the four members of the fact team associated with the pair.

Multiplication and Division Tables

Write the following on the board:



With a piece of colored chalk in each hand, place one piece on the 1 in the row headed by 1 and the other piece on the 1 in the column headed by 1. Move the first piece horizontally to the right, while

the other is moved vertically down so that the pieces meet in a cell. When the two meet, ask, "What does 1 times 1 equal?" Write 1 in that cell.

Now place the first piece of chalk on the 1 in the row headed by 1 and the second piece on the 2 in the column headed by 2. Move the first piece horizontally to the right and the other vertically down. Ask, "What is 1 times 2?" Record the answer in the cell where the two pieces of chalk meet. Proceed to fill in the table so that eventually all the products the pupils have explored to date are entered. A sketch of such a completed table looks like this:

X	0	1	2	3	4	5	6	7	8	9	10
0											
1		1	2	3	4	5	6	7	8	9	10
2		2	4	6	8	10					
3		3	6	9							
4		4	8								
5		5	10								
6		6									
7		7									
8		8									
9		9									
10		10									

The table can be used as a division table by reversing the procedure as follows: Place two pieces of chalk on any one of the cell entries. Move one piece of chalk horizontally to the left to the number heading the row, as follows:

(÷) X	0	1	2	3	4	5	6	7	8	9	10
0											
1		1	2	3	4	5	6	7	8	9	10
2		2	4	6	8	10					
3		3	6	9							
4		4	8								
5		5	10								
6		6									
7		7									
8		8									
9		9									
10		10									

Ask, "Eight divided by four equals what number?" Pupils will tell you to move the other piece of chalk vertically upwards to the number (2) at the head of the column to find the answer. Work out enough division situations so that everyone knows how to use the multiplication table as a division table.

Start the class off on building other "Times Tables" by having them place one rod of each color on their desks. Have them cross each rod with a white rod. Write the following on the board:

One Times Table
1 x 1 = ___
1 x 2 = ___
1 x 3 = ___
1 x 4 = ___
1 x 5 = ___
1 x 6 = ___
1 x 7 = ___
1 x 8 = ___
1 x 9 = ___
1 x 10 = ___

Have them complete the table as you record the answers. To start building the "two times table," have them place a white rod, a red rod, a light green rod, a purple rod, and a yellow rod on their desks and cross each with a red rod. Since the class explored only products less than or equal to ten, you might tell the class that the restriction on other crosses is a temporary one. Have the pupils complete the table. Proceed until all times tables from one to ten are begun and completed as far as their experiences permit. A summary of their tables looks like the drawings below and on the next page:

One Times Table
1 x 1 = 1
1 x 2 = 2
1 x 3 = 3
1 x 4 = 4
1 x 5 = 5
1 x 6 = 6
1 x 7 = 7
1 x 8 = 8
1 x 9 = 9
1 x 10 = 10

Two Times Table
2 x 1 = 2
2 x 2 = 4
2 x 3 = 6
2 x 4 = 8
2 x 5 = 10
2 x 6 = ___
2 x 7 = ___
2 x 8 = ___
2 x 9 = ___
2 x 10 = ___

Three Times Table
3 x 1 = 3
3 x 2 = 6
3 x 3 = 9
3 x 4 = ___
3 x 5 = ___
3 x 6 = ___
3 x 7 = ___
3 x 8 = ___
3 x 9 = ___
3 x 10 = ___

Four Times Table
4 x 1 = 4
4 x 2 = 8
4 x 3 = ___
4 x 4 = ___
4 x 5 = ___
4 x 6 = ___
4 x 7 = ___
4 x 8 = ___
4 x 9 = ___
4 x 10 = ___

Five Times Table
5 x 1 = 5
5 x 2 = 10
5 x 3 = ___
5 x 4 = ___
5 x 5 = ___
5 x 6 = ___
5 x 7 = ___
5 x 8 = ___
5 x 9 = ___
5 x 10 = ___

Six Times Table
6 x 1 = 6
6 x 2 = ___
6 x 3 = ___
6 x 4 = ___
6 x 5 = ___
6 x 6 = ___
6 x 7 = ___
6 x 8 = ___
6 x 9 = ___
6 x 10 = ___

Seven Times Table
7 x 1 = 7
7 x 2 = ___
7 x 3 = ___
7 x 4 = ___
7 x 5 = ___
7 x 6 = ___
7 x 7 = ___
7 x 8 = ___
7 x 9 = ___
7 x 10 = ___

Eight Times Table
8 x 1 = 8
8 x 2 = ___
8 x 3 = ___
8 x 4 = ___
8 x 5 = ___
8 x 6 = ___
8 x 7 = ___
8 x 8 = ___
8 x 9 = ___
8 x 10 = ___

Nine Times Table
9 x 1 = 9
9 x 2 = ___
9 x 3 = ___
9 x 4 = ___
9 x 5 = ___
9 x 6 = ___
9 x 7 = ___
9 x 8 = ___
9 x 9 = ___
9 x 10 = ___

Ten Times Table
10 x 1 = 10
10 x 2 = ___
10 x 3 = ___
10 x 4 = ___
10 x 5 = ___
10 x 6 = ___
10 x 7 = ___
10 x 8 = ___
10 x 9 = ___
10 x 10 = ___

The Sum and Product Game

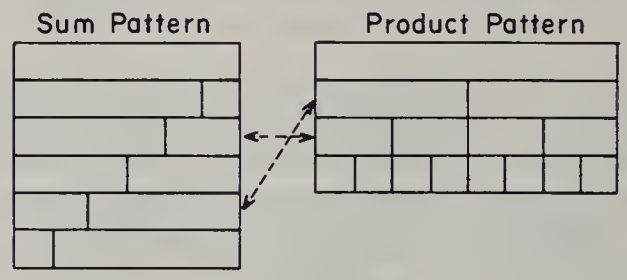
Have pupils place a red rod and a yellow rod on their desks. Have them make a train and find the rod whose length is equal to that of the train. Now have them form a cross with the red and yellow rods. Have them construct the floor suggested by the cross, rearrange the floor into a train, and find the rod whose length is equal to that of the train. Write on the board:

Sum: 7
Product: 10

Tell the class you are thinking of two numbers. They can find these numbers just by knowing the clues written on the board. Most pupils will relate your clues to their actions upon the rods to form the sum 7 and the product 10. They will quickly tell you that your numbers must be 2 and 5, since:

$2 + 5 = 7$ and $2 \times 5 = 10$ or $5 + 2 = 7$ and $5 \times 2 = 10$

Ask the class to construct with their rods the addition pattern (two-car train) for 6 and the multiplication pattern for 8. Their constructions look like this:



Then write the following:

Sum: 6
Product: 8

Ask which pair of rows would be selected, one from each pattern, so that they represent both a sum of six and a product of eight. (There are two correct answers.) Have them select two rods such that a train of these rods and a cross of the same two rods would suggest the two rows they would select from their patterns. Pupils will scan each pattern and most likely select a red rod and a purple rod to represent

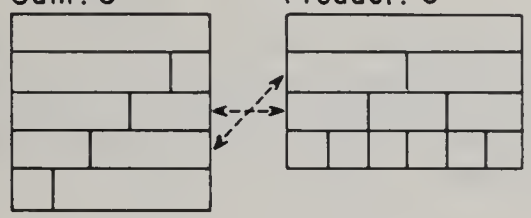
$4 + 2 = 6$ and $4 \times 2 = 8$ or $2 + 4 = 6$ and $2 \times 4 = 8$

Guide the class through several more Sum and Product games before preparing Ditto sheets with four or more games per sheet for two-member teams to work on. A sample of such Ditto sheets might contain the following problems:

Sum : 5 Product : 6	Sum : 4 Product : 3
Sum : 8 Product : 7	Sum : 6 Product : 9

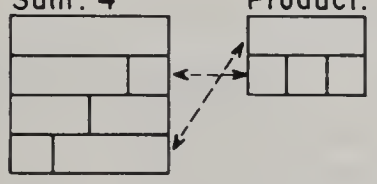
Pupil teams experiencing any difficulties should be encouraged to construct the train patterns for the sums and products to assist them in completing each game. A summary of the rod patterns associated with the sample Ditto sheet above looks like this:

Sum : 5 Product : 6



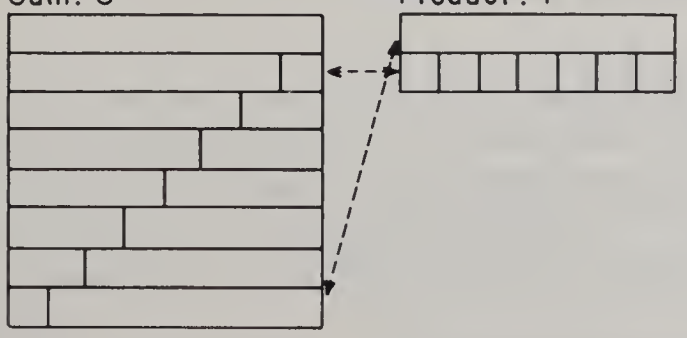
3 + 2 = 5 or 2 + 3 = 5
3 x 2 = 6 2 x 3 = 6

Sum : 4 Product : 3



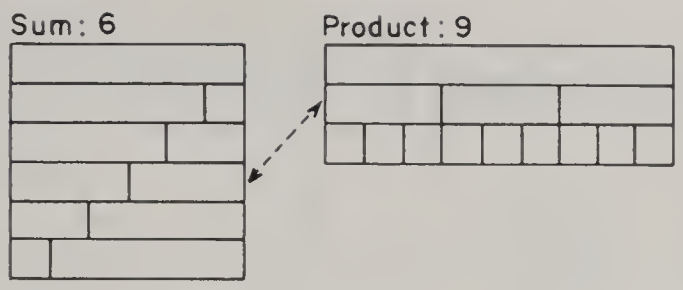
3 + 1 = 4 or 1 + 3 = 4
3 x 1 = 3 1 x 3 = 3

Sum : 8 Product : 7



7 + 1 = 8 or 1 + 7 = 8
7 x 1 = 7 1 x 7 = 7

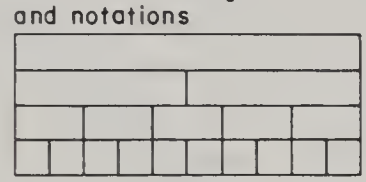
Sum : 6 Product : 9



3 + 3 = 6 , 3 x 3 = 9

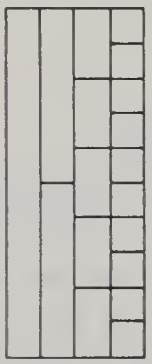
Note: All horizontal notations should also be written in the vertical form. This can be readily done by having pupils construct a product pattern in the horizontal form and then having them rotate the pattern row by row one-quarter of a complete rotation and record the results in the vertical form. Examples of the rod patterns and notations look like this:

Horizontal arrangement and notations

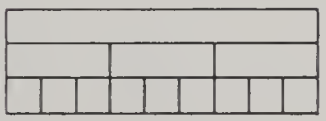


1 x 10 = 10
2 x 5 = 10
5 x 2 = 10
10 x 1 = 10

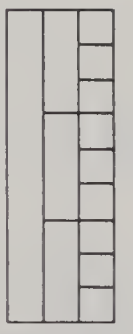
Vertical arrangement and notations



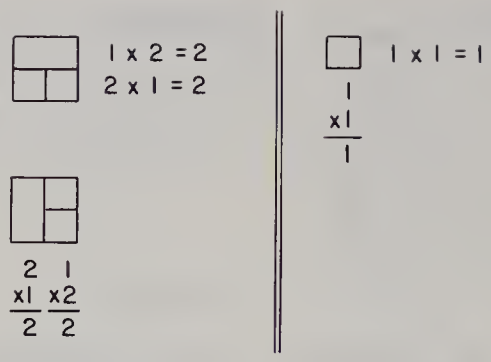
10 5 2 1
x1 x2 x5 x10
10 10 10 10



1 x 9 = 9
3 x 3 = 9
9 x 1 = 9



9 3 1
x1 x3 x9
9 9 9



The Missing Factor Game

Have the class place a red rod and an orange rod on their desks. Write the following on the board:

$$\begin{array}{r} \square \\ \times 2 \\ \hline 10 \end{array} \qquad 10 \div 2 = \square$$

Tell the class the orange rod represents a product (10) and the red rod one of the factors (2) of the product. Have them find the missing factor. Form the cross, floor, and train, if necessary, to verify the result. Ask what number should be written in the box and complete the writings as directed by the pupils:

$$\begin{array}{r} \boxed{5} \\ \times 2 \\ \hline 10 \end{array} \qquad 10 \div 2 = \boxed{5}$$

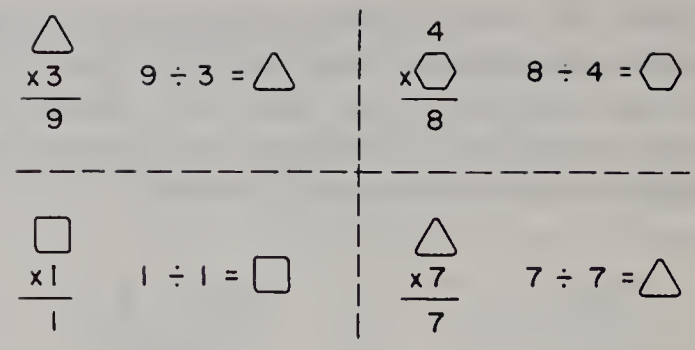
Now have each pupil place an orange rod and a yellow rod on his desk. Tell the class the orange rod represents the product 10 and the yellow rod one of the factors of 10. Have them find the other factor (forming cross, floor, and train, if necessary). Write the following on the board:

$$\begin{array}{r} \square \\ \times 5 \\ \hline 10 \end{array} \qquad 10 \div 5 = \square$$

And complete the sentence:

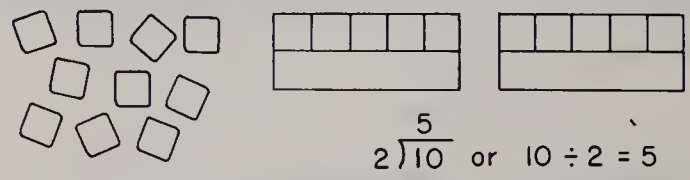
$$\begin{array}{r} \boxed{2} \\ \times 5 \\ \hline 10 \end{array} \qquad 10 \div 5 = \boxed{2}$$

Prepare Ditto sheets with four or more games per sheet. Two-member teams can be formed to work on the Ditto sheets. Any disagreement can be settled by resorting to the rods for verification. A sample Ditto sheet looks like this:

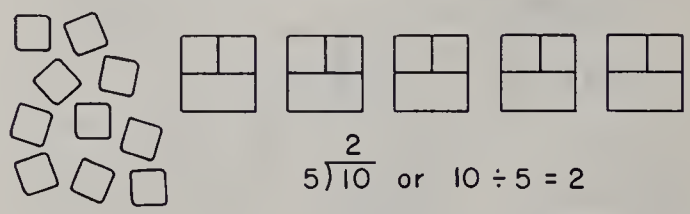


The Partitioning Game

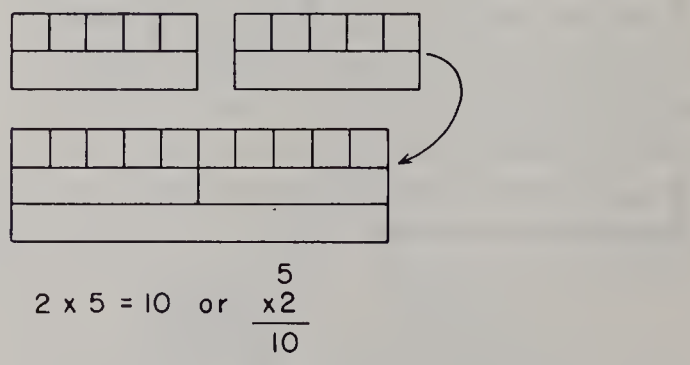
Have the class place ten white rods on their desks. Ask pupils to partition (separate) the white rods into two sets so that each set has the same number of white rods in it. Have them verify their partitioning by placing a rod equal in length to each train of white rods alongside of each train as shown below:

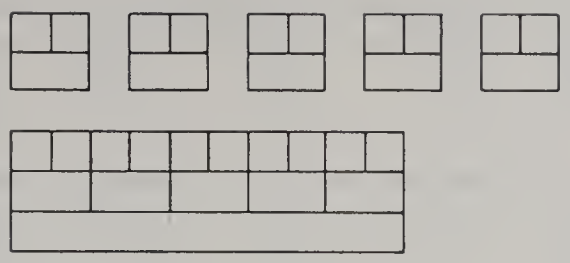


Have them once again place the ten white rods on their desks. Tell them they are to form five groups equal in number and verify their partitioning as shown below:



After each partitioning is constructed and the number sentence describing their actions is completed, pupils should reconstruct the original situation and write the multiplication sentences describing the reconstruction. For example, in the two previous partitioning situations the reconstructions would look like this:

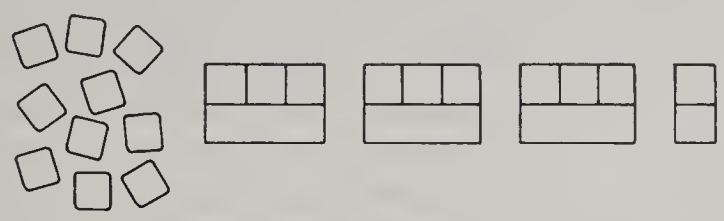




$5 \times 2 = 10$ or $\begin{array}{r} 2 \\ \times 5 \\ \hline 10 \end{array}$

A situation like the following should also be explored before pupils form two-member teams for playing partitioning games: "Place ten white rods on your desks. Can you partition your ten white rods into three groups equal in number? Try to do so!"

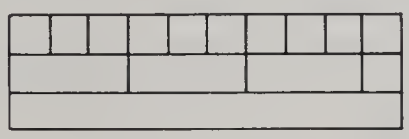
Pupils will construct the following:



and tell you they can construct three groups equal in number with one white rod left over. Record the information as follows:

$3\overline{)10}$ 3 rem. 1

Have them reconstruct the original situation. A sketch and associated sentence describing the reconstruction would be:



$3 \times 3 + 1 = 10$

Have the class form two-member teams to play their own partitioning and reconstruction games.

A selects a number of white rods and specifies the partition he wishes B to construct.

B constructs the partitioning and records his actions.

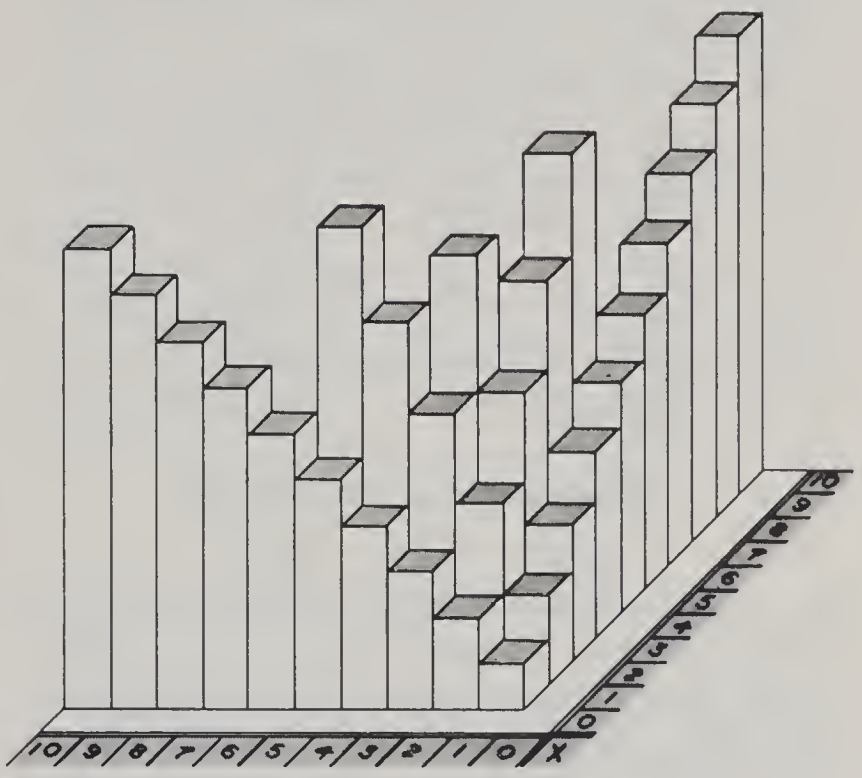
A reconstructs B's partitioning and records his actions.

And so on.

Multiplication Tables with Rods

Prepare Ditto sheets or use graph paper with a centimeter grid for the multiplication table constructions.

Have pupils complete the multiplication table by writing in all the cell entries for products from 1 through 10. Have pupils stand up rods (where possible) on the cells, with the rods representing the products in the cells. A sketch of their construction looks like this:



Some other tables for pupils to construct are suggested below:

X	0	1	2	3	4	5	6	7	8	9	10
10		10									
9		9									
8		8									
7		7									
6		6									
5		5	10								
4		4	8								
3		3	6	9							
2		2	4	6	8	10					
1		1	2	3	4	5	6	7	8	9	10
0											

X	0	2	4	6	8	10
1		2	4	6	8	10
3		6				
5		10				
7						
9						

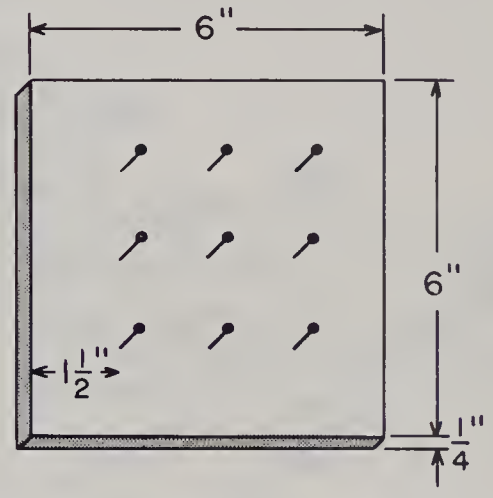
X	2	3	5	7
2	4	6	10	
3	6	9		
5	10			
7				

X	1	3	5	7	9
1	1	3	5	7	9
3	3	9			
5	5				
7	7				
9	9				

NONMETRIC GEOMETRY WITH GEO-BOARDS

(Properties of geometric figures not involving measurements)

The models we are discussing in this section are called *Geo-boards*, a name coined by Caleb Gattegno. Geo-boards are boards that pupils can use to explore geometry. Here is a picture of a Geo-board.

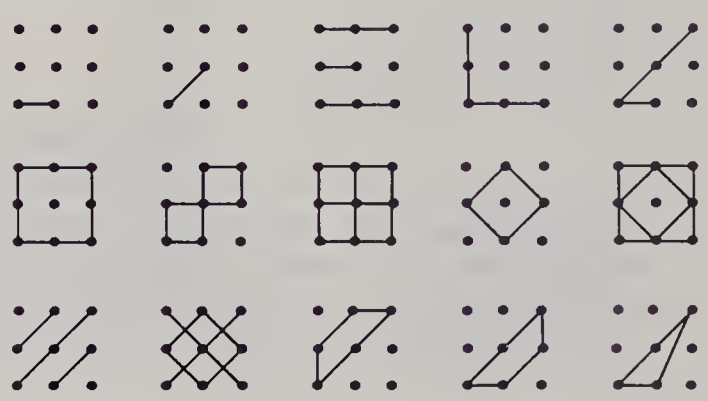


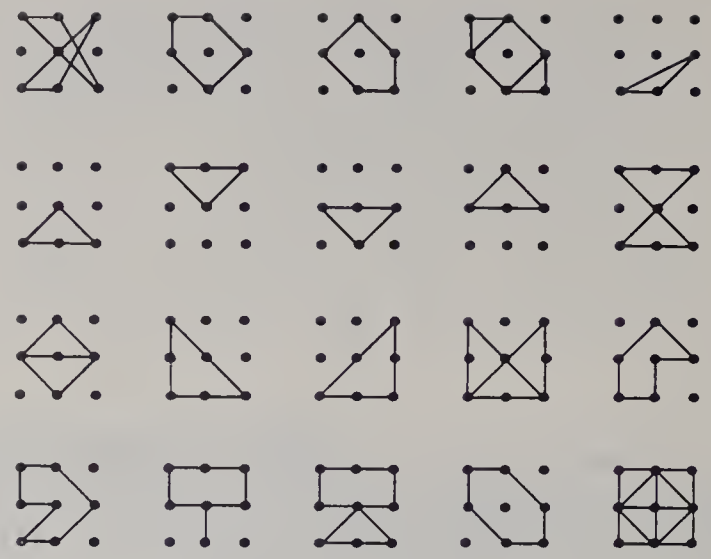
It is a 6-inch by 6-inch by ¼-inch plywood or pine wood board with a square array of ½-inch brads.

The "9-point Geo-boards," rubber bands of various colors, and a piece of string are all the physical equipment needed for the activities which follow. The Industrial Arts department of the senior or junior high school can easily and cheaply construct 9-point Geo-boards for each of the pupils in your class.

Free-play Activities

Distribute a Geo-board, some rubber bands of various colors, and a piece of string to each pupil. Retain one set of these materials for yourself. Tell the pupils they will have several sessions of free play with their equipment. The following is a sample of pupil constructions:





The above patterns clearly demonstrate how quickly pupils in a relaxed and natural atmosphere of free play will introduce themselves to a variety of geometric figures. Here, too, as with the Cuisenaire® rods, pupils are participating observers constructing, decomposing, and otherwise altering their patterns to please themselves or copying a neighbor's pattern because it strikes their fancy. The careful, observant teacher tunes in on the spontaneous language pupils use to describe constructions to their peers while making mental notes of such language. Examples are: point, line, flat, three-, four-, five-sided pictures, squares, boxes, diamond, and so on.

Representations and Descriptions of Points

You are the best judge of when to begin structuring the free play sessions. Whenever this decision is made, consider questions leading to game-like activities similar to those suggested throughout the remainder of this section.

"Who can show us a *point?*"

"Everyone place one finger on any point on your own Geo-board."

"Johnny, look at the point your finger is on."

"Jane, look at the point your finger is on."

"Are you both touching the same point or are you touching different points?"

"How many points are being touched by the class?" (If there are twenty-five pupils and yourself, then a careful count will show twenty-six points being touched.)

"What if each of us touched two points—how many points in all would we be touching?" (Lots!)

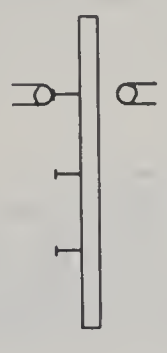
"Suppose that every pupil in each of the classes

of our school was touching a point. Now how many points would be touched?" (A whole bunch of points!)

"What if we filled the room with Geo-boards and tagged a point on each one of them. How many points would be tagged? (A whole bunch of points!)

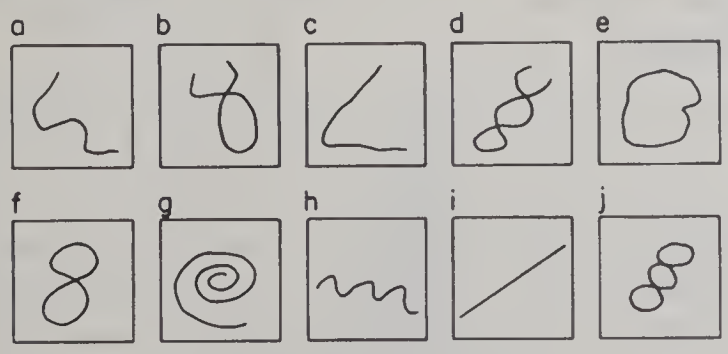
"What if we were to tag a point on each of enough Geo-boards to fill the whole school with them. How many points would we have to tag?" (Wowie!)

Have the class turn their Geo-boards over on the other side. If anyone has a nail punched through onto this side, get rid of this Geo-board until it can be repaired. "Touch one of your points on the other side and point with a finger from your other hand to where the point would be located on this side of the board." Do the same thing for another point . . . and another. Are there lots of points on both sides of the board? Are there lots of points on both sides of everyone's board?



Representations and Descriptions of Curves

Have the class place their Geo-boards flat on their desks so the side with no nail protruding is up. Distribute a piece of string to each pupil. With the board placed on their desks as suggested, tell the class that they can have some free play time with the board and the string. Most pupils are puzzled by this at first, but soon begin to construct figures like this:



Sketch situations like those above so questions of this kind can be asked and answered:

"Is (a) a *curve?*"

"Is (b) a curve? (c)? (d)? . . ."

"Is (i) a curve?" (Perhaps a *straight curve!*)

"Which of your curves are *open curves?*" (a, b, c, d, g, h, i)

"Which of your curves are *closed curves?*" (e, f, j)

"Of your closed curves, which is the simplest closed curve?" (e)

"Why is (e) the simplest closed curve?" (no cross-over points)

"How many cross-over points in (f)?" (one)

"How many cross-over points in (j)?" (two)

"Who can construct a closed curve with three cross-over points? four? five? zero?"

Another way to describe a *simple closed curve* would be to say that it is a closed curve with zero cross-over points.

"Turn your Geo-boards over so that the nail side is up. With your string, construct a simple closed curve that has only one of the nails showing in its interior."

"Construct a simple closed curve that has two nails in the interior and seven in the exterior."

Construct a simple closed curve having three nails showing in its interior, three on the curve, and three in its exterior."

"Construct a simple closed curve with nine nails showing in its interior, zero nails on the curve, and zero nails in the exterior."

Points as Locations in Space

Have several pupils located in widely separated parts of the room stand up and face each other. Have one hold his Geo-board over his head, another hold his Geo-board at his feet, another hold his board upside down, another hold his board tilted up, etc. Have each place a finger on one of the nail heads.

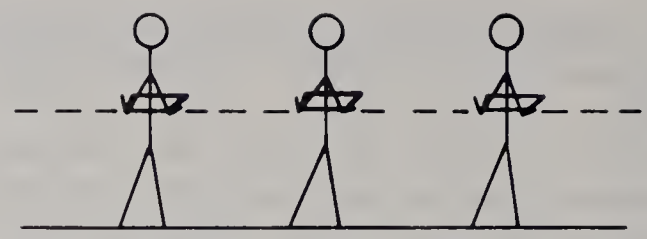
"Jane, show the class where your point is *located*. Jim, where is your point located? John, where is your point located?" And so on.

The idea we wish to establish by the above activities is that points are locations in space, and that space is the set of all points. Some further activities to help establish this idea might be as follows:

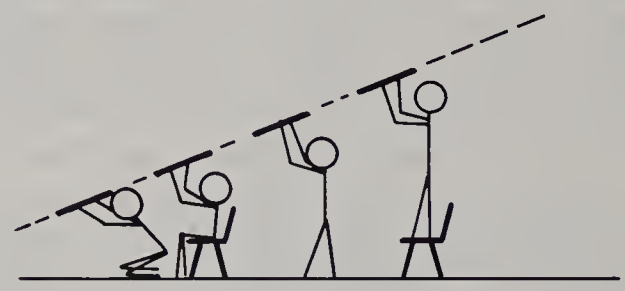
Walk around the room holding your Geo-board and touching one of the nail heads. Pause and ask, "As I moved from here to here, what remained the same?" You are the same person, you are holding the same Geo-board, you are touching the same nail head. What changed? Your position in the classroom changed and the position of your Geo-board and the position or location of the nail head you are touching changed. Stand in front of the room and move your Geo-board into many positions. Have members of the class walk around the room. Have several leave the room. Frequently ask, "What remains the same? What changes?" Now ask, "How many points did we locate?" (Quite a few!) "If we had enough time, could we locate many more points in the space of this room? How about the space outside the room? Is a *point* a *location in space?*" (Yes!) "Can we think of *space* as being the set of all *possible points* or locations?" (Yes!)

Representations and Descriptions of Planes

Have pupils stand holding their Geo-boards over their heads and looking at each other. "Are all the points (represented by the nail heads) level?" Hold your Geo-board at a convenient height, parallel to the floor, and ask the class to hold their Geo-boards at the same level. Walk around the room assisting pupils to line their points up with those on their neighbors' boards. Ask the class what name they could give to the set of all points in space that are at the same level as their points seem to be at.



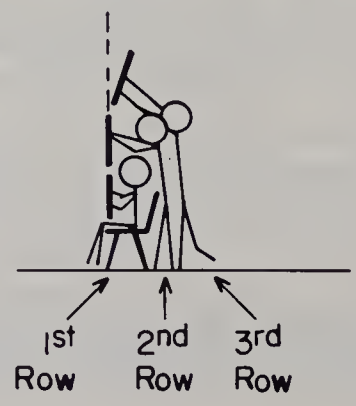
Ask the class to lift their boards a little higher, a little lower, each time trying to keep at the same level as demonstrated by you. Have half the class line up into five or six rows to tilt their Geo-boards so as to represent a tilted plane as shown below:



Have members of the other half of the class attempt to describe this situation. Then have this latter half line up in five or six rows, tilting their boards the other way:

The rest of the class attempts to describe this situation.

Draw a line on the floor at the front of the room. Have one group of pupils line up along the line holding their Geo-boards in front of them. A second row lines up behind them, but lines its Geo-boards up with the first row and slightly higher up. A third row (complications) attempts to line its boards up with the other two rows and slightly higher. The rest of the class tries to describe this situation.



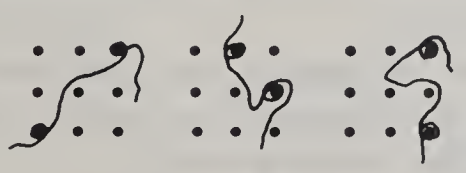
Each of the situations described and sketched was designed to get at the notion of a *plane* as being a set of points and represented by a flat surface.

Put some dots on the chalkboard representing the nail heads of Geo-boards. Ask if all the points on the chalkboard are in the same plane. How about all the points on the floor? The ceiling? The left wall? The right wall? the rear wall? We shall call the set of points suggested by a flat surface a *plane*.

Have the class place their Geo-boards on their desks. Ask if anyone can represent more points than those represented by the nine nails, still keeping the Geo-boards on the desk. If no one does, suggest the following. Place a finger on the center nail head and rotate the Geo-board. Spinning the Geo-board about the center point will suggest the movement of the nails to different locations. Thus, the nine nail heads of their Geo-board represent only nine of the many points in the plane of the Geo-board. If more nails were driven into the board, more points could be represented, and so on.

Paths of Points

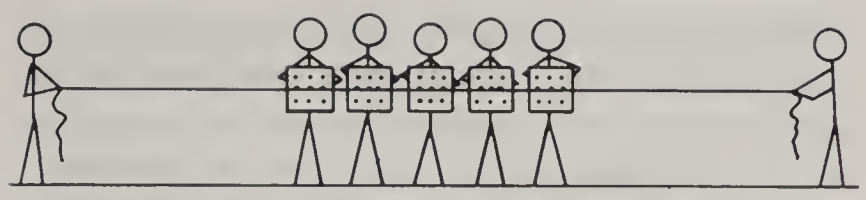
Have pupils loop their piece of string around any two nails of their Geo-boards and let the rest of the string lie on the board. Sketch some of their constructions on the board as follows:



Have pupils come to the board to trace the *path of points* in one of your sketches, while others run their fingers along their paths of points.

Straight Lines

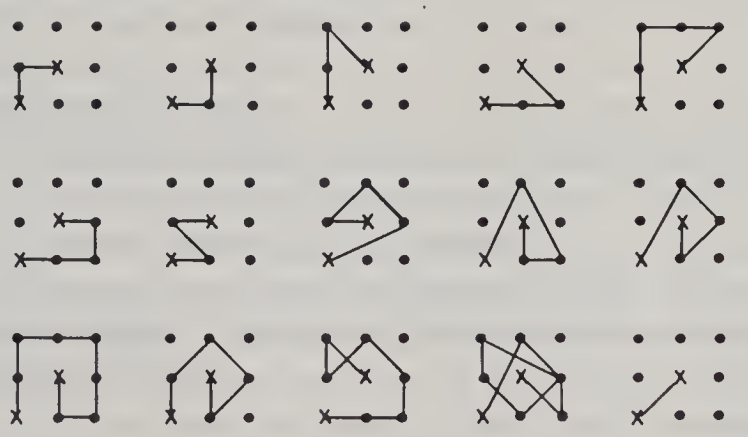
Locate two pupils on opposite sides of the room. Holding a string long enough for them to hold it taut (no slack), ask a pupil to line up one row of nails of his Geo-board with the string. Have two more locate themselves on either side of the first pupil and line up one of their rows of nails. Continue to have pairs of pupils line up on either side of the preceding pairs, so that eventually all pupils have lined up one row of their Geo-board nails with the string as sketched below:



Have the pupils swing their Geo-boards to and fro, keeping the one row of nail heads in contact with the stretched string to note that, although the other nail heads on the Geo-board move to different locations, those touching the string remain in the same position. Ask if anyone knows what the string represents. Of course, what is being represented is a set of points called a *straight line*. Have the pupils holding the string take steps to the left and right, respectively, while those with the lined-up Geo-boards attempt to keep their positions on the line. Ask if their actions suggest a set of lines in a plane. Have all the pupils move forward and backward, keeping their Geo-boards lined up with the string. Ask if they are once again representing a set of lines in a plane. "If we had enough string and enough pupils holding Geo-boards, would the straight line ever end?" (No!)

Line Segments

Tell the class to pick two points on their plane (nail heads on the Geo-board) and use rubber bands to construct different paths of points to join the two points they selected. Some possibilities might be:

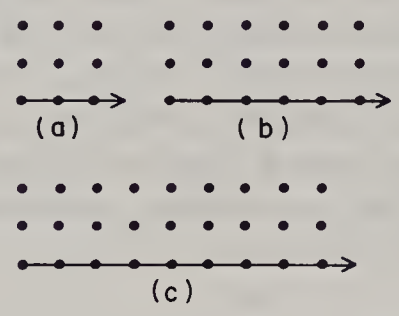


"Which path of points is the simplest, most direct, straightest, shortest path of points? What name could we give this kind of a path of points? Is it a line? Does it have a beginning point? An ending point? Both a beginning and ending point? Does a line have beginning and ending points, or does it go on and on in both directions? Is it a piece or segment of a line? We call this kind of

path of points or piece of a line a *line segment*. A *line segment*, then, is the set of all points on a line between two points on the line and including these two points. Construct as many line segments as you can with your rubber bands and Geo-boards." Some sketches of the constructions look like this:

Rays

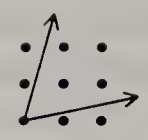
"Could we have a path of points which is part of a line so that the path has a beginning point but no ending point? How could we represent such a path of points?" One way might be as follows:



Hook a rubber band onto one point (nail) and stretch it over several more and beyond the edge of the Geo-board to suggest the starting point but no ending point as in (a). Have two pupils stand in front of the class. Tie a piece of string onto one of the nails of one of the Geo-boards. Pass the string over several more points and beyond the edge of the first board onto and beyond the edge of the second one as in (b). Invite a third pupil to line up some of his points with the string as in (c). Continue this procedure for several more pupils with Geo-boards. If our string were long enough and if we had all the pupils we wished with Geo-boards to hook onto our string, would there be any end to our construction? No! This representation, then, is what we mean by a *ray*—a subset of a line which consists of a starting point on the line and all the points on the line to one side of the starting point.

Angles

Have the pupils form two-member teams, each team working with one Geo-board. Each team member is to construct a ray, using the same starting point. See the sketch in the next column:



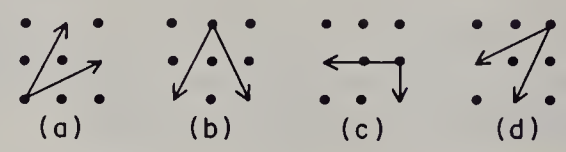
Ask how they can simplify matters so that each can work from his own Geo-board. A possible solution would be to allow each to tentatively hook each rubber band to another nail with the agreement that, although the rubber bands look like line segments, they can be extended when required.

"Construct two rays so that the starting point of each is the same point. Construct several more such figures." A sketch of some of the constructions might be:

"Who knows what name we give to sets of points like these?" (Angles) An angle, then, is the set of points on two rays not on the same line but with a common starting point.

"Construct some more angles. Put your finger on a starting point of any angle. We call this point of the angle the *vertex* of the angle. The *rays* of the angle are often called the *sides* of the angle. Pluck some of the sides (rubber bands) of some of your angles so that we all know what we mean by the sides of an angle.

"Construct an angle so that there are two nails showing in the interior of the angle." Several possibilities might be the following:



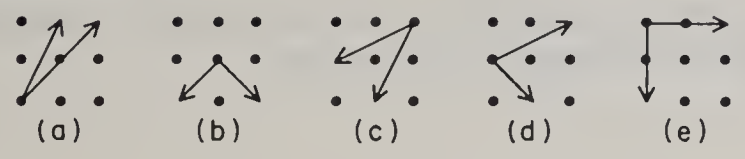
In (a) above, if there are two nails showing in the interior of the angle, how many are showing in the exterior of the angle? (4) In (b), how many are showing in the exterior? (4) In (c)? (3) In (d)? (4)

"Construct some angles so that there will be zero nails showing in the interior." Some possibilities are:

"Construct several angles so that there are zero nails showing in the exterior of the angle." Some possibilities are:

"Construct five angles, one at a time, so that:

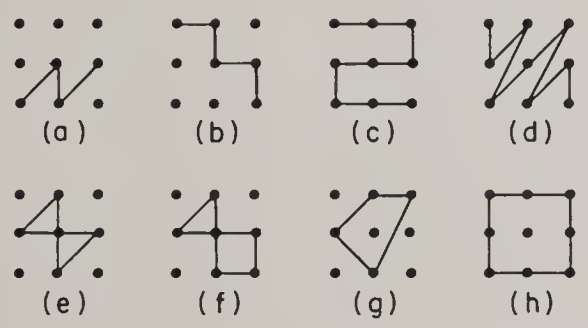
a) zero nails are showing in the interior,
b) one nail is showing in the interior,
c) two nails are showing in the interior,
d) three nails are showing in the interior,
e) four nails are showing in the interior."



Similar construction problems can be posed for nails showing on the angle as well as nails showing in the exterior.

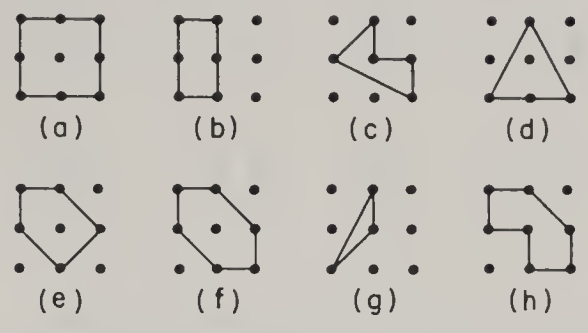
Polygons

Ask pupils to construct what they think is a broken-line curve. Of course, what we are after initially is a string of line segments placed end to end so that at least one line segment contains one point not on the same line as the other line segments of the set. A sketch of some of the possibilities looks like this:



Sketch some of the pupil constructions on the board, including some of those suggested above. "Which of these sketches of your constructions represent broken line curves?" (All) "Which represent open curves?" (a, b, c, d) "Which represent closed curves?" (e, f, g, h) "Of those representing closed curves, which are not simple closed curves?" (e, f) "Which are the simple closed curves?" (g, h)

"We call simple closed curves constructed with line segments *polygons*. Construct some polygons." Some possibilities might be:



"Construct some polygons so that there are zero nails showing in the interior." (b, c, g, and h show polygons which meet the requirements)

"Construct some polygons so that there is just one nail showing in the interior." (a, d, e, f meet these requirements)

"Construct a polygon so that there is only one nail showing in its interior and zero nails showing in its exterior." (a meets the requirements)

"If there are nine points on your Geo-board and your polygon shows one nail in its interior and zero nails in its exterior, where are the other eight nails showing?" (They are on the polygon.)

"Construct a polygon that has zero nails showing in its interior and two nails showing in its exterior." (h meets these requirements)

"Construct a polygon so that there is one nail in its interior and three in its exterior." (e)

Obviously there are many opportunities for similar questions to be asked.

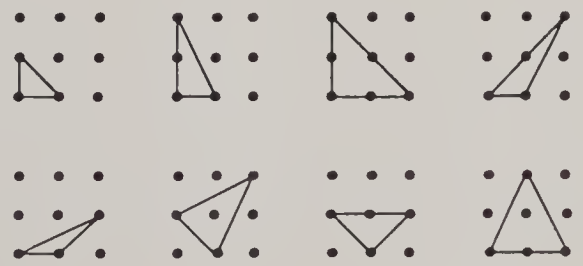
Classifying Polygons

"Construct a one-sided polygon." (Can't do it!)

"Construct a two-sided polygon." (Can't do it!)

"Construct a three-sided polygon."

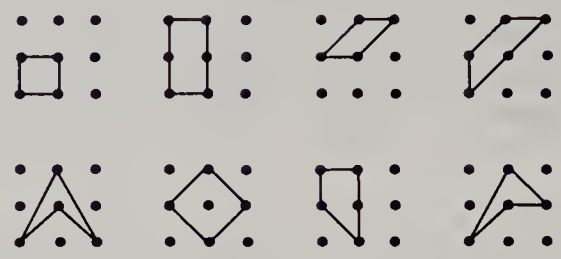
"Construct as many different looking three-sided polygons as you can." Some possibilities are sketched below:



We call three-sided polygons *triangles*.

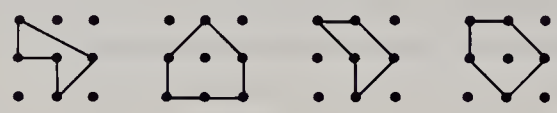
"Construct as many different looking four-sided

polygons as you can." Some possibilities are sketched below:



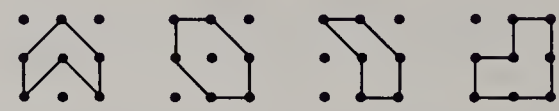
We call four-sided polygons *quadrilaterals*.

"Construct some five-sided polygons." Some possibilities are sketched below:



We call five-sided polygons *pentagons*.

"Construct some six-sided polygons." Some possibilities are:



We call six-sided polygons *hexagons*.

In all of the preceding discussion, the attempt has been to offer suggested activities aimed at exposure and not at closure or mastery. Pupils need much more and varied experiences before we can expect closure on many of the fundamental nonmetric properties of the elementary geometric figures considered in this section.

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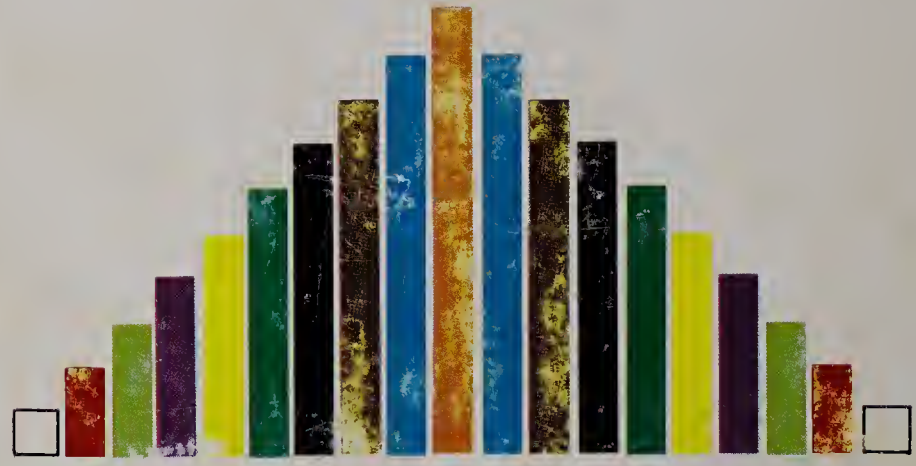

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